

Summary: Properties of a Family of Random Self-Similar Iterated Function Systems on the Line

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The dissertation considers a special family of statistically self-similar one-dimensional sets. Prominent examples of this family are rational projections of random sponges. Here a sponge refers to a self-similar fractal set constructed by recursively dividing the space into a uniform grid of side length $1/L$ aligned with the coordinate axes, and selectively retaining specific sub-regions. The objective of the research is to explore how the Hausdorff dimension and Lebesgue measure of the obtained random attractor change with the variation of probability parameters, and whether it contains interior points.

The formulation of the question dates back to Marstrand's projection theorem. According to this theorem, if the dimension of a planar Borel set is strictly greater than one, then almost all of its projections have positive length, whereas if the dimension is less than one, the projection does not reduce the dimension of the set in almost every direction. For a certain family of random sets, the Mandelbrot percolation set, an even stronger statement holds. In two dimensions, it is constructed as follows: We fix an integer $L > 1$ and a probability $0 < p \leq 1$. The closed unit square is divided into L^2 congruent sub-squares, each of which is retained independently with probability p and discarded with probability $1 - p$. This process is repeated in the remaining squares indefinitely, or until no squares are left. The set thus obtained is called the Mandelbrot percolation set; for a choice of $p > 1/L^2$, it is a non-empty set with positive probability, whose dimension is $\log(L^2 p)/\log(L)$ almost surely, provided that the set is non-empty. In the case of Mandelbrot percolation, if the dimension of the set is greater than one, then the projection contains an interval almost surely, provided that the set is non-empty, simultaneously for projections in all directions. If the dimension is less than or equal to one, then the dimension does not drop. In contrast, several examples found in the literature show that if the percolation process is restricted to the building blocks of planar (non-trivial) sponges, a completely different picture emerges. The aim of my dissertation is to refine this picture: to investigate a broader family than those found in the literature (allowing for more complex overlap systems than previously known), and to comprehensively study their properties. The main results of the dissertation can be summarized as follows:

1. *Positivity of Lebesgue measure:* The dissertation provides a sharp condition for when the random attractor has positive Lebesgue measure, using the Lyapunov exponent corresponding to the matrices describing the system. If the Lyapunov exponent is positive, the measure is positive; if it is negative, or in certain critical cases, the measure is zero.
2. *Condition for missing interior points:* We prove that if the lower spectral radius of the matrices describing the system is strictly less than one, then the interior of the attractor is almost surely empty, even in the case of a positive Lebesgue measure.
3. *Positive Lebesgue measure, empty interior:* We study the conditions under which the random attractor has positive Lebesgue measure and empty interior. We provide a partial solution to this using the convexity of the subadditive pressure function corresponding to the matrices describing the system.
4. *Interior points:* We provide a sufficient condition for the attractor to almost surely contain an interior point.
5. *Dimension-theoretic results:* In the dissertation, we determine the dimension of the random attractor using the subadditive pressure function corresponding to the matrices describing the system.

The dissertation illustrates the developed theory through concrete examples, such as the 45-degree projection of the right-angled Sierpiński carpet.