# Properties of a Family of Random Self-Similar Iterated Function Systems on the Line

Thesis booklet

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# 1 Introduction

In this thesis booklet we give a short summary of the motivation of the thesis, state the main theorems of the dissertation and show some applications. The dissertation is about a special family of statistically self-similar sets.

The Mandelbrot percolation fractal in $\mathbb{R}^2$ is constructed inductively as follows. We fix an integer $L \geqslant 2$ and a probability $p \in (0, 1)$. The closed unit square is divided into $L^2$ congruent sub-squares, each of which is independently retained with probability $p$ and discarded with probability $1 - p$. This process is repeated in the retained squares *ad infinitum*, or until there are no squares left. The Mandelbrot percolation process restricted to the building blocks of a sponge yields a random sponge. We consider the rational projections of random sponges[1] into lines from the perspective of Hausdorff dimension, positivity of Lebesgue measure and existence of interior points. In contrast to Mandelbrot percolation, the inhomogeneity of the sponge yields qualitatively different phenomena.

A well-known example of a 2-dimensional sponge is the Sierpiński-carpet, however in this thesis booklet we focus on the much simpler *right-angled Sierpiński carpet*, for the construction of this see Section 4.

## 1.1 Motivation of the study

An early, influential theorem of projection theory is Marstrand's projection theorem from 1954 [15].

**Theorem 1.1.** *Let* $E \subset \mathbb{R}^2$ *be a Borel set. Then the following holds.*

- *If* $\dim_{\mathrm{H}} E > 1$*, then almost every projection of* $E$*, to lines, has positive Lebesgue measure.*

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[1] by rational projections we mean orthogonal projections to lines, with rational tangents. By sponges we mean self-similar sets, for which the first level cylinders are aligned to a $1/L$-grid for some $2 \leqslant L \in \mathbb{N}$.

- *If* $\dim_{\mathrm{H}} E > 1$ *then almost every projection of* $E$*, to lines, has the same Hausdorff dimension as* $E$*.*

There is a blooming afterlife of this theorem, we only mention some, which are related to certain random sets. Let $\mathcal{M}_p = \mathcal{M}_p^L$ denote the Mandelbrot percolation with parameters $L$ and $p$. It is well known (see for example Kahane [13], Hawkes [11], Falconer [6], , Mauldin-Williams [16]) that in this case

$$\dim_{\mathrm{H}} \mathcal{M}_p = \frac{\log L^2 p}{\log L} \text{ almost surely conditioned on } \mathcal{M}_p \neq \varnothing^{2}. \tag{1.1}$$

The following theorem shows that for the Mandelbrot percolation a stronger version of Marstrand's theorem holds.

**Theorem 1.2** (Rams-Simon, [22], [23])**.**

- *If* $\dim_{\mathrm{H}} \mathcal{M}_p > 1$ *then almost surely conditioned on non-extinction, the projection to* all *lines simultaneously* contains an interval *almost surely.*
- *If* $\dim_{\mathrm{H}} \mathcal{M}_p \leqslant 1$ *then almost surely the projections to all lines simultaneously have Hausdorff dimension equal to* $\dim_{\mathrm{H}} \mathcal{M}_p$*.*

The reason why the above can hold for every *direction* is the spatial homogeneity of the Mandelbrot percolation fractal. It was shown by Simon and Vágó in [25], that if we run an analogous percolation process on the building blocks of the Sierpiński carpet, then for the resulting random set Theorem 1.2 does not hold.

**Theorem 1.3** (Simon-Vágó [25])**.** *Let* $\mathcal{S}_p$ *denote the random Sierpiński carpet with parameter* $p$*. For every angle* $\alpha$ *with rational tangent, there exists a parameter interval* $I_\alpha$*, so that for* $p \in I_\alpha$ *simultaneously* $\dim_{\mathrm{H}} \mathcal{S}_p > 1$ *but* $\mathrm{proj}_\alpha \mathcal{S}_p$ *has empty interior almost surely.*

We note that this is overall not so surprising, because already in the deterministic case it is well known that for sponges rational directions are usually exceptional.

It is natural to ask what behaviours can we observe as we vary the probability parameter $p$ in case of rational projections of random *inhomogeneous sponges* such as the random Sierpiński carpet.

**Coordinate projections** A very well-understood case is when one considers projections to coordinate axes. This has been studied by Falconer, Dekking, Grimmet, Meester, from the perspective of Hausdorff and box dimensions [3, Theorem 9] and [7], existence of interior points [8], positivity of Lebesgue measure [3, Theorem 8]. We remark that these results holds for a more general family of examples than how we describe them here. These results are in terms of the expectations $\{m_i\}_{i=0}^{L-1}$, namely let $m_i$ be the expected number of first level cylinders that project to the $i$-th $L$-adic interval:

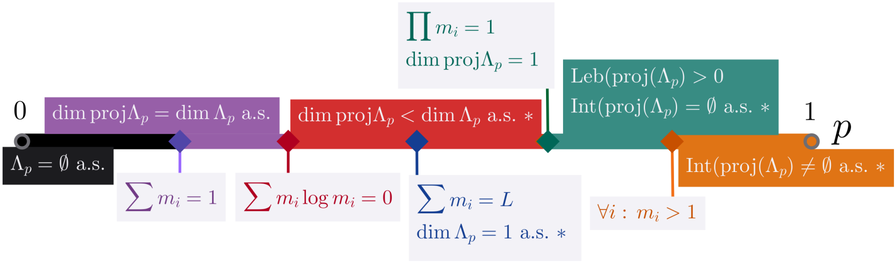

Figure 1: The parameter intervals and their boundaries as we vary the probability parameter $p$ in case when we project a random sponge $\Lambda_p$ to the coordinate axis. The projected set is denoted by $\operatorname{proj}\Lambda_p$, $m_i$ are the column-wise expectations, and $*$ denotes conditioned on non-extinction.

$[iL^{-1}, (i+1)L^{-1}]$. Then under some weak regularity conditions the following holds for the projected random set $\operatorname{proj}\Lambda$:

1. If $m_i > 1$ for all $i$, then $\mathrm{Int}(\operatorname{proj}\Lambda) \neq \varnothing$ conditioned on non-extinction. (Here Int stands for the interior of the set.)
2. If $\prod_{i=0}^{L-1} m_i > 1$ then $\operatorname{proj}\Lambda$ has positive Lebesgue measure almost surely conditioned on non-extinction,
3. $\dim_{\mathrm{H}} \operatorname{proj}\Lambda = \dim_{\mathrm{B}} \operatorname{proj}\Lambda = \inf_{t\in[0,1]} \log(\sum_{i=0}^{L-1} m_i^t)/\log(L)$.

The above together with some consequences is summarized in Figure 1 in the case when not all $m_i$ is the same. **Sums of random Cantor sets** We only mention here [4, 5, 17], where the authors considered differences of 1-dimensional random Cantor sets. In particular they studied the existence of interior points and positivity of Lebesgue measure of these sets. The connection to our topic is coming from the fact that the difference of the set is a rescaled version of the 45-degree projection of the product set. The proof ideas and techniques introduced in these papers are heavily utilized in our work as well.

**Further notation** Infinite words and matrices are denoted by boldface letters, and vectors and finite words are underlined. For $K \in \mathbb{N}$ we use $[K] = \{0, \dots, K-1\}$, and $[K]_1 = \{1, \dots, K\}$.

For the alphabet $\mathcal{I}$ we denote by $\mathcal{I}^n$ the set of words of length $n$, $\mathcal{I}^* = \bigcup_{n\in\mathbb{N}} \mathcal{I}^n$ the set of finite words, and by $\Sigma^{\mathcal{I}}$ the set of infinite words over $\mathcal{I}$. Let $\underline{\theta}|_n$ and $\boldsymbol{\theta}|_n$ denote the prefix of length $n$ of the finite word $\underline{\theta}$ and the infinite word $\boldsymbol{\theta}$. For a non-negative matrix $\mathbf{M}$ let $M(i,j)$ denote the $j$-th element of the $i$-th row, and $\|\mathbf{M}\| = \sum_i \sum_j M(i,j)$.

**Summary** In this thesis, we investigated whether the properties observed for axes-projections (as illustrated in Figure 1) apply to general rational projections. Initially, it was unclear if similar intervals exist or if new parameter intervals would emerge. Ultimately, our current results suggest that no new phenomena occur.

# 2 Setup

In this section, we introduce a family of deterministic IFSs, called the integer self-similar IFSs, and the stochastic randomization process used to select which elements to retain and discard, ultimately generating a random attractor. A finite list of contractions $\mathcal{S} := \{S_i(x)\}_{i \in [M]_1}$ is called a *contracting iterated function system* (IFS). In this thesis we confine ourselves to *homogeneous, one-dimensional, self-similar* IFSs, where each function in the IFS is a similarity: namely, for each $i \in [M]_1$, the map $S_i : \mathbb{R} \to \mathbb{R}$ has the form $S_i(x) = rx + t_i$ for some $r \in (-1, 1)\backslash\{0\}$ and $t_i \in \mathbb{R}$.

We use the notational shorthand

$$S_{i_1 \dots i_n} = S_{i_1} \circ \cdots \circ S_{i_n}.$$

There exists a compact set $B \subset \mathbb{R}$, for which $S_i(B) \subset B$. By a classical result of Hutchinson, for this IFS $\mathcal{S}$, there exists a unique non-empty compact set satisfying

$$\Lambda := \bigcup_{i \in [M]_1} S_i(\Lambda).$$

We refer to this set as the *attractor*, and note that it can be equivalently defined as

$$\Lambda = \bigcap_{n \in \mathbb{N}} \bigcup_{(i_1, \dots, i_n) \in [M]_1^n} S_{i_1 \dots i_n}(B).$$

## 2.1 Integer iterated function systems

In this dissertation we require the following two properties to hold for the IFS. Examples of such IFSs include rational projections of 2-dimensional sponges.

**(a)** All contractions are the reciprocal of the same integer: that is $r = 1/L$ for an integer $L \geqslant 2$.

**(b)** All translations $t_i$ are rational numbers.

These IFSs can be conjugated to a self-similar IFSs with common contraction ratio $1/L$ and translations $0 = t_1 \leqslant t_2 \leqslant \cdots \leqslant t_M$ where $t_i \in \mathbb{Z}$ for all $i$ and $L - 1$ divides $t_M$. We call IFS of this form a *Integer Self-Similar IFS* (ISSIFS). The cylinder sets of these IFSs contain significant overlaps. Generally, overlaps can make the analysis very complicated. The reason for considering these IFSs comes from the fact that their overlapping structure can be described using a finite set of matrices.

We fix an integer self-similar IFS $\mathcal{S}$, with common contraction ratio $1/L$, and $\#\mathcal{S} = M$. Let $\eta$ be the self-similar measure corresponding to the IFS $\mathcal{S}$ and the probability vector $(1/M, \ldots, 1/M)$: for Borel sets $A$, the measure $\eta$ satisfies

$$\eta(A) = \sum \frac{1}{M} \eta(S_j^{-1} A).$$

Next, we introduce a family of partitions (mod 0) of $\mathbb{R}$ and refer to the elements of these partitions as *$L$-adic intervals*:

$$\mathcal{D}_k := \left\{ \left[ (i-1)L^{-k}, iL^{-k} \right] : i \in \mathbb{Z} \right\}, \quad k \in \{-1, 0, 1, 2, \ldots\}. \tag{2.1}$$

Particular attention is given to those elements of $\mathcal{D}_{-1}$ which have positive $\eta$-measure. We call them *basic intervals*. We let $N$ denote the number of basic intervals, and the basic intervals are

$$\left\{ J^{(u)} := [b_u L, (b_u + 1)L] \right\}_{u=1}^{N}, \quad b_u \in \mathbb{N},\ b_u < b_{u+1},\ u = 1, \ldots, N-1.$$

The smallest interval that contains all the basic intervals is

$$I = \left[ 0, L \frac{t_M}{L-1} \right]. \tag{2.2}$$

For every $k \in [N]_1$ and $n \in \mathbb{N}$ the basic interval $J^{(k)}$ subdivides into $L^n$ congruent subintervals of length $L^{-(n-1)}$ contained in $\mathcal{D}_{n-1}$, which we denote by $J^{(k)}_{\underline{\theta}}$, where $\underline{\theta} = (\theta_1, \ldots, \theta_n) \in [L]^n$. More precisely, for $u \in [N]_1$ and $\underline{\theta} = (\theta_1, \ldots, \theta_n) \in [L]^n$:

$$J^{(u)}_{\underline{\theta}} = \left[ b_u L + \sum_{\ell=1}^{n} \theta_\ell L^{-(\ell-1)}, b_u L + \sum_{\ell=1}^{n} \theta_\ell L^{-(\ell-1)} + L^{-(n-1)} \right].$$

A key property of these IFSs, is the following matrix representation of cylinder counts, which was introduced by Victor Ruiz in [24]. For every $\underline{\theta} \in [L]^n$ for $n \geqslant 1$ we define the $N \times N$ matrix

$$B_{\underline{\theta}}(u, v) := \# \left\{ \underline{\ell} \in [M]_1^n : S_{\underline{\ell}}(J^{(v)}) = J^{(u)}_{\underline{\theta}} \right\}.$$

It can be shown that for $\underline{\theta} = (\theta_1, \ldots, \theta_n) \in [L]^n$ for $n > 1$,

$$B_{\underline{\theta}}(u, v) = (\mathbf{B}_{\theta_1} \cdots \mathbf{B}_{\theta_n})(u, v).$$

## 2.2 The randomization

**Definition 2.1** (Substitution self-similar sets)**.** Let $\mathcal{S} := \{S_i\}_{i=1}^{M}$ be an integer self-similar IFS and let $\underline{p} = (p_h)_{h \in \mathcal{P}([M]_1)}$ be a probability distribution on the power set of $[M]_1$. The corresponding substitution self-similar set $\Lambda$ is defined as follows: In the first step, we pick a random subset $\mathcal{E}_1 \subset [M]_1$ such that $\mathbb{P}(\mathcal{E}_1 = h) = p_h$ for any $h \in \mathcal{P}([M]_1)$. Assume that the set of retained nodes at level $n$, denoted $\mathcal{E}_n \subset [M]_1^n$, has already been constructed. For every retained node $\mathtt{i} \in \mathcal{E}_n$, we choose (independently) a random subset $\mathcal{E}_1^{[\mathtt{i}]} \subset [M]_1$ having the same distribution $\underline{p}$ as $\mathcal{E}_1$. The set of offspring for a retained node $\mathtt{i}$ is defined by $O(\mathtt{i}) = \{\mathtt{i}k \in [M]_1^{n+1} : k \in \mathcal{E}_1^{[\mathtt{i}]}\}$.

We then define the level $n+1$ retained nodes as the union of all offspring:

$$\mathcal{E}_{n+1} = \bigcup_{\mathtt{i} \in \mathcal{E}_n} O(\mathtt{i}) \subset [M]_1^{n+1}.$$

Finally, the substitution self-similar set is defined by

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{\mathtt{i} \in \mathcal{E}_n} S_{\mathtt{i}}(I),$$

for $I$ defined in (2.2).

In the applications we consider a special family, which we refer to as *coin tossing IFS*. In this case we fix one probability parameter $p \in (0,1]$, and we run the percolation process on the cylinder sets. This is the special case, when $\underline{p} = (p_h)_{h \subset [M]_1}$ is given by

$$p_h = p^{\#h}(1-p)^{M-\#h}. \tag{2.3}$$

## 2.3 An important quantity: The expectation matrices

The results are stated in terms of the expectation matrices. These are the expectation analogue of the matrices $\{\mathbf{B}_{\underline{\theta}}\}_{\underline{\theta} \in [L]^*}$. Namely, let $\{\mathbf{M}_{\underline{\theta}}\}_{\underline{\theta} \in [L]^*}$ be the matrices satisfying:

$$M_{\underline{\theta}}(u,v) = \mathbb{E}\left(\# \left\{\underline{\ell} \in [M]_1^n : S_{\underline{\ell}}(J^{(v)}) = J_{\underline{\theta}}^{(u)}\right\}\right). \tag{2.4}$$

for $\underline{\theta} \in [L]^n$.

We remark, that for $(\theta_1, \dots, \theta_n) = \underline{\theta} \in [L]^n$ $\mathbf{M}_{\underline{\theta}} = \mathbf{M}_{\theta_1} \cdots \mathbf{M}_{\theta_n}$, and that for coin tossing systems $\mathbf{M}_{\underline{\theta}} = p^n \mathbf{B}_{\underline{\theta}}$.

# 3 Results and applications

The dissertation is based on the following papers: [18–20]. These are all accepted to journals and are based on joint work with Károly Simon. Theorem 3.2, 3.6 and Lemma 3.7, 3.8 appeared in [20]. There, they were proven in higher dimension, for coin-tossing systems. Theorem 3.12 appeared in [20] in this form. Its general version, Theorem 3.11 together with Theorem 3.15 appear in this dissertation first.

In this section, we state each result for substitution self-similar sets systems corresponding to a 1-dimensional integer IFS. The IFS is $\mathcal{S} = \{S_i(x) = x/L + t_i\}_{i=1}^{M}$, the attractor is denoted by $\Lambda$ (with $\Lambda_p$ reserved for coin tossing systems with probability parameter $p$). We emphasize that the common contraction ratio is $L^{-1}$, the number of maps is $M$, and there are $N$ basic intervals denoted by $J^{(1)}, \dots, J^{(N)}$.

The expectation matrices are $\mathcal{M} = \{\mathbf{M}_0, \dots, \mathbf{M}_{L-1}\}$. These matrices are non-negative by construction. We say that a non-negative matrix is *allowable* if each row and column of each expectation matrix contains a strictly positive element.

**Definition 3.1** (jointly positively irreducible matrices)**.** Consider the set of non-negative, allowable matrices $\{\mathbf{M}_0, \dots, \mathbf{M}_{L-1}\}$. Given a probability measure $\nu$ on $\Sigma^{[L]}$ we say that the matrices are *jointly positively irreducible* if there exists a finite word $(\theta_1, \dots, \theta_n) \in \Sigma_n^{[M]_1}$ such that $\mathbf{M}_{\theta_1} \dots \mathbf{M}_{\theta_n}$ is a strictly positive matrix, and $\nu([\theta_1, \dots, \theta_n]) > 0$. When the measure is implicit, we mean that the matrices are jointly positively irreducible with respect to the uniform measure.

## 3.1 Ergodic measures on the random attractor

The theorem appearing here is based on the work [19], and appeared in [20] for coin tossing systems. The contributions to the field are as follows:

- In the dissertation we prove [21, Theorem 2.6] regarding the extinction of multitype branching processes in random environments assuming that the expectation matrices only satisfy a weak positivity condition. Results of a similar flavour exist in the literature (see for example [1, 14, 26], or the survey [27]), but only with strong positivity assumptions on the expectation matrices. These assumptions rarely hold for expectation matrices coding random self-similar sets with overlaps.

- Using [21, Theorem 2.6] we extend the results on sharp conditions for which a random set can have positive measure for a fixed ergodic measure (for example the Lebesgue measure) from sets with only exact or negligible overlaps to integer IFSs. (Theorem 3.2.)

### Concepts

Let $\Pi_L \colon \Sigma^{[L]} \to [0,1]$ denote the standard $L$-adic coding map and, for a basic interval $J^{(u)}$, we define the projection $\Gamma_u \colon [0,1] \to J^{(u)}$ for $x \in [0,1]$ by $\Gamma_u(x) = L(b_u + x)$. We consider the composition projection $\Gamma_u \circ \Pi_L \colon \Sigma^{[L]} \to J^{(u)}$. We write

$$\widetilde{\nu} := \sum_{j=1}^{N} (\Gamma_u \circ \Pi_L)_* \nu, \tag{3.1}$$

where $(\Gamma_u \circ \Pi_L)_* \nu$ denotes the pushforward measure of $\nu$ with respect to the map $\Gamma_u \circ \Pi_L$. The quantity

$$\lambda_\nu(\mathcal{M}) := \lim_{n\to\infty} \frac{1}{n} \frac{1}{L^n} \sum_{\underline{\theta} \in [L]^n} \log \|\mathbf{M}_{\underline{\theta}}\|,$$

is called the *Lyapunov exponent with respect to $\nu$*. When the measure is not specified, we refer to the uniform measure on $\Sigma^{[L]}$. Equivalently, $\lambda_\nu(\mathcal{M}) = \lim_{n\to\infty} 1/n \log(\|\mathbf{M}_{\boldsymbol{\theta}|_n}\|)$ for $\nu$-almost every $\boldsymbol{\theta} \in \Sigma^{[L]}$, where $\boldsymbol{\theta}|_n$ stands for the prefix of $\boldsymbol{\theta}$ of length $n$.

The main result of this section, Theorem 3.2, is closely related to the extinction of certain branching processes. In order to treat the critical case, we require an additional regularity condition, which we now describe.

We define the vector-valued random variables $\underline{Y}(\theta, v) = (\#\{i \in \mathcal{E}_1 : S_i(J^{(w)}) = J_\theta^{(v)}\})_{w\in[N]_1}$, for $\theta \in [L]$ and $v \in [N]_1$.

**Condition 1.** For a fixed ergodic measure $\nu$ on $\Sigma^{[L]}$ there exists $\theta \in [L]$ so that $\nu([\theta]) > 0$ and for each $v \in [N]_1$:

$$\mathbb{P}(\underline{Y}(\theta, v) \in \{\underline{0}\} \bigcup \bigcup_{j=1}^{N} \underline{e}_j) < 1.$$

### Results

**Theorem 3.2.** *Consider an ergodic measure $\nu$ on $\Sigma^{[L]}$. If $\mathcal{M}$ is jointly positively irreducible with respect to $\nu$, then the following hold for $\widetilde{\nu}$ defined in* (3.1)*:*

- *If $\lambda_\nu(\mathcal{M}) > 0$ then $\widetilde{\nu}(\Lambda) > 0$.*
- *If $\lambda_\nu(\mathcal{M}) < 0$ then $\widetilde{\nu}(\Lambda) = 0$.*
- *If $\lambda_\nu(\mathcal{M}) = 0$ and the vector random variables $\{\underline{Y}(\theta, v)\}$ corresponding to the random attractor satisfy Condition 1 then $\widetilde{\nu}(\Lambda) = 0$.*

**Applications**

- **Positive Lebesgue measure in case of coin tossing systems.** For the coin tossing system with parameter $p$ we consider the set of matrices $\mathcal{B} = (\mathbf{B}_\theta)_{\theta \in [L]}$ corresponding to the deterministic system. Assume that they are jointly positively irreducible. Then

  $$\mathrm{Leb}(\Lambda_p) > 0 \text{ iff } p > \exp(-\lambda(\mathcal{B})),$$

  where $\lambda(\mathcal{B})$ is the Lyapunov expoennt corresponding to the set of matrices $\mathcal{B}$. Typically, it is very difficult to estimate the Lyapunov exponent corresponding to a set of matrices.

  *Application* 3.3. Pollicott and Vytnova proved (this is based on personal communication) that for the 45-degree projection of the right-angled Sierpiński carpet, the lower and upper bounds for the Lyapunov exponent $\lambda(\mathcal{B})$ are 0.3961 and 0.3962 respectively. Hence, for $p > 0.673$, $\mathrm{Leb}(\Lambda_p) > 0$ almost surely conditioned on non-extinction.

## 3.2 Non-existence of interior points and positive Lebesgue measure

The contributions can be summarized as follows:

- We provide a theorem on non-existence of interior points in the random attractor for integer IFSs. This can be considered as an extension of the result in [8, Theorem 1], which provides a sharp condition on the non-existence of interior points for IFSs with exact or negligible overlaps. We note that sharp result formulated in terms of the lower spectral radius appeared in [4] for the difference of two random Cantor sets, which is the 45-degree projection of the product of two random Cantor sets.
- In the case of a coin tossing integer IFS, we give conditions under which there exists a non-trivial interval $(p_0, p_1)$ of probability parameters such that for any $p_0 < p < p_1$ the random attractor $\Lambda_p$ has positive Lebesgue measure but empty interior almost surely conditioned on non-extinction.

In order to discuss these two concepts simultaneously, we introduce the notion of the *subadditive pressure function.* This encodes information about the two quantities, as its right-derivative at zero is precisely the Lyapunov exponent and its asymptote at minus infinity upper bounds the logarithm of the lower spectral radius.

**Notions**

**Definition 3.4.** The *interesting parameter interval* is an open interval $(p_0, p_1)$ with $0 \leqslant p_0 < p_1 \leqslant 1$ such that for every $p \in (p_0, p_1)$ the random (coin-tossing)

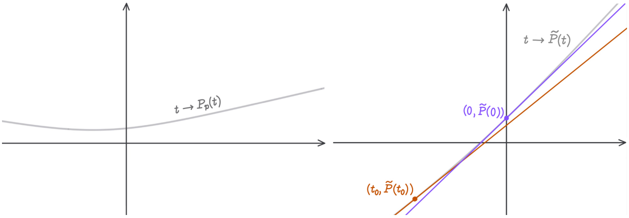

Figure 2: The first figure depicts the properties of the subadditive pressure corresponding to the expectation matrices $\mathcal{M}$ (when the interesting parameter exists). The second figure shows the tangents of the subadditive pressure corresponding to the matrices $\mathcal{B}$ and the tangents at zero and a point $t_0 < 0$.

attractor $\Lambda_p$ has empty interior almost surely, but has positive Lebesgue measure almost surely conditioned on non-extinction.

**Definition 3.5** (Lower spectral radius)**.** Fix a set of non-negative matrices $\mathcal{M} = \{\mathbf{M}_0, \dots, \mathbf{M}_{L-1}\}$. We define

$$\breve{\rho}_n(\mathcal{M}, \|\cdot\|) := \inf\{\|\mathbf{M}_{i_1} \cdots \mathbf{M}_{i_n}\|^{\frac{1}{n}},\ \mathbf{M}_{i_j} \in \mathcal{M},\ 1 \leqslant j \leqslant n\}.$$

Then, the *lower spectral radius* of the matrices $\mathcal{M}$ is given by

$$\breve{\rho}(\mathcal{M}) = \lim_{n\to\infty} \breve{\rho}_n(\mathcal{M}, \|\cdot\|).$$

This limit exists by subadditivity and Fekete's Subadditive Lemma (for more on the lower spectral radius see [12]). Moreover, it is intuitively clear (and explicitly proved in [12]) that $\breve{\rho}(\mathcal{M})$ is bounded above by the spectral radius of any single $n$-fold product of matrices raised to the power $1/n$. From the Theorem 3.6 we can conclude that under the joint positivity condition on the expectation matrices, if $\lambda = \lambda_\nu(\mathcal{M}) > 0$ but $\log(\breve{\rho}(\mathcal{M})) < 0$, then the attractor has positive Lebesgue measure but empty interior (almost surely conditioned on non-extinction).

We now introduce the *subadditive pressure function*:

$$P(t) := \lim_{n\to\infty} \frac{1}{n} \log\Big( \sum_{\underline{\theta}\in[L]^n} \|\mathbf{M}_{\underline{\theta}}\|^t \Big). \tag{3.2}$$

Properties of the subadditive pressure are proved in [9, 10]. The relevant properties are that for $\mathcal{M}$, $P(t)$ exists for all $t \in \mathbb{R}$, and it is convex and continuous. Moreover, it is continuously differentiable for $t > 0$.

In particular, $\lim_{t\to 0+} P'(t) = \lambda$ and $\lim_{t\to-\infty} P(t)/t \geqslant \log(\breve{\rho}(\mathcal{M}))$. It follows that if $P(t)$ is strictly convex on a subinterval of $(-\infty, 0]$, then $\log(\breve{\rho}(\mathcal{M})) < \lambda$.

### Results

**Theorem 3.6.** *Suppose the expectation matrices satisfy* $\check{\rho}(\mathcal{M}) < 1$. *Then the interior of the attractor is almost surely empty.*

**Lemma 3.7.** *Suppose that* $\mathcal{M}$ *is jointly positively irreducible. If* $P(t)$ *is strictly convex on a subinterval of* $(-\infty, 0)$, *then the interesting parameter interval exists.*

We mention one final result here, which is a corollary of more general theorem of Tom Rush which we can apply in some special cases.

**Lemma 3.8** (Corollary of a Theorem of Tom Rush)**.** *Assume that* $\mathcal{M}$ *consists of invertible matrices and*

- *there is* $\mathbf{M} \in \mathcal{M}$ *whose eigenvalues are all distinct in norm;*
- *and there exists* $\widetilde{\mathbf{M}} \in \mathcal{M}$ *such that* $\widetilde{\mathbf{M}}(V) \cap W = \{0\}$ *for any pair of* $\mathbf{M}$*-invariant subspaces with complementary dimensions.*

*Then the pressure is either affine on its entire domain or strictly convex in a neighbourhood of* $0$.

### Applications

- **Coin tossing systems** In case of coin tossing systems the pressure can be written as
  $$P_p(t) = \widetilde{P}(t) + t\log(p),$$
  where $\widetilde{P}(t)$ is pressure corresponding to the matrices $\{\mathbf{B}_\theta\}_{\theta\in[L]}$ of the deterministic IFS. If for $\{\mathbf{B}_\theta\}_{\theta\in[L]}$, the pressure satisfies the strict convexity condition mentioned in Lemma 3.7, then it is easy to see that for $p \in (e^{-\lambda(\mathcal{B})}, \check{\rho}(\mathcal{B}))$ the random attractor has positive Lebesgue measure but empty interior almost surely conditioned on non-extinction.

- **Some specific examples of coin tossing systems**

  *Application* 3.9. For some particular coin tossing systems, e.g. in case of Example 4.1, there are estimates of the Lyapunov exponent and the lower spectral radius that proves the existence of the parameter interval, where the interesting parameter interval exists.

  *Application* 3.10. An easy calculation shows that the assumptions of Lemma 3.8 are satisfied for Example 4.2. The fact that the pressure is not affine near one in case of this example is proved by Bárány and Rams in [2, Proof of Theorem 1.3]. Consequently, in this case the interesting parameter interval exists.

## 3.3 Existence of interior points

The contributions to the field:

- We prove a theorem (Theorem 3.11) on existence of interior points in the random attractor. This is an extension of similar results concerning sets with negligible or exact overlaps to general integer IFSs. We mention that a sharp result in terms of the lower spectral radius appears in [4] for the difference of two random Cantor sets.
- We provide examples (Application 3.14) where using the *Wolfram Mathematica* programming language, we estimated the probability parameter where almost surely (conditioned on non-extinction) the attractor contains an interval.

### Results

**Theorem 3.11.** *Suppose the expectation matrices* $\mathcal{M}$ *are allowable, and moreover that there exists a set of vectors* $\mathcal{U} = \{\underline{u}_1, \dots, \underline{u}_\ell\}$ *with* $\underline{u}_i = (u_i(1), \dots, u_i(N)) \in$ $\mathbb{N}^N \backslash \{\underline{0}\}$ *such that the following hold:*

1. *There exists* $v \in [N]_1, S^* \in \mathbb{N}, \underline{u}^* \in \mathcal{U}$ *and* $\underline{\theta} \in [L]^{S^*}$ *so that*

   $$\mathbb{P}(\underline{Y}(\underline{\theta}^*, v) \geqslant \underline{u}^*) > 0,$$

   *for the vector valued random variables* $\underline{Y}(\underline{\theta}, v) = (\#\{\underline{i} \in \mathcal{E}_n : S_{\underline{i}}(J^{(w)}) = J_{\underline{\theta}}^{(v)}\})_{w \in [N]_1}$*, for* $\underline{\theta} \in [L]^n$ *and* $v \in [N]_1$*.*

2. *There exists* $R \in \mathbb{N}$ *and* $\gamma > 1$ *and* $h^* \subset [M]_1$ *with* $p_h > 0$*, such that for all* $\underline{\theta} \in [L]^R$ *and for all* $\underline{u} \in \mathcal{U}$ *there exists a* $\underline{v} \in \mathcal{U}$ *with*

   $$\underline{u}^T \mathbf{B}_{\underline{\theta}}^{(\underline{h^*})^R} \geqslant \gamma \underline{v}, \text{ and } \underline{u}^T \mathbf{M}_{\underline{\theta}} \geqslant \gamma \underline{v},$$

   *where* $(\underline{h^*})^R = (h^*, \dots, h^*)$*.*

*Then the random attractor* $\Lambda$ *contains an interval almost surely, conditioned on non-extinction.*

### Applications

**Coin tossing systems** For coin tossing systems, Theorem 3.11 simplifies to Theorem 3.12.

**Theorem 3.12.** *Consider the coin tossing self-similar integer IFS with expectation matrices* $\mathcal{M} = \{\mathbf{M}_0 = p \cdot \mathbf{B}_0, \dots, \mathbf{M}_{L-1} = p \cdot \mathbf{B}_{L-1}\}$*. Assume that there exists a non-empty set* $\mathcal{U} := \{\underline{u}_1, \dots, \underline{u}_m\}$ *with* $\underline{u}_i = (u_i(1), \dots, u_i(N)) \in \mathbb{N}^N \backslash \{\underline{0}\}$ *such that the following hold:*

1. *There exists* $\underline{u}^* \in \mathcal{U}$ *and* $\underline{\theta}^* \in [L]^{S^*}$ *for some* $S^* \geqslant 1$*, and* $U^* \in [N]_1$ *such that*
$$\underline{e}_{U^*}^T \mathbf{B}_{\underline{\theta}^*} \geqslant \underline{u}^*. \tag{3.3}$$
2. *There exists a* $\gamma > 1$ *and a level* $R$ *such that for all* $\underline{u} \in \mathcal{U}$ *and for all* $\underline{\theta} \in [L]^R$ *there exists a* $\underline{v} \in \mathcal{U}$ *such that*
$$\underline{u}^T \mathbf{M}_{\underline{\theta}} \geqslant \gamma \underline{v}.$$

*Then* $\Lambda_p$ *contains an interval almost surely, conditioned on non-extinction.*

**Corollary 3.13.** *With the setup as in Theorem 3.12, assume that the following hold:*

- *there exists a* $\theta \in [L]$ *such that* $\mathbf{B}_\theta$ *has a strictly positive row, and*
- *for all* $\theta \in [L]$ *the matrix* $\mathbf{M}_\theta$ *have all column sums greater than one.*

*Then the random attractor contains an interval almost surely conditioned on non-extinction.*

*Application* 3.14 (Example 4.2). In the case of Example 4.2 (which is the "0-1-3 problem", with contraction ratio 1/2),

$$\mathcal{U} = \{(1,0,1),(1,1,0),(0,1,1)\}$$

and we estimated the optimal $\gamma$ at level 20, which gave us the following result: for $p > 463^{-1/20} \sim 0.7357$, $\Lambda_p$ contains an interval almost surely conditioned on non-extinction.

## 3.4 Dimension theory of the random attractor

### Notions

In this result we again rely on the subadditive pressure function defined for matrix-product (see (3.2)). The contributions of this part are as follows:

- We provide a theorem concerning the value of the almost sure dimension of the random attractor. This is an extension of similar results for sets with either negligible or exact overlaps to case of integer IFSs. (Theorem 3.15.)

### Results

**Theorem 3.15.** *Fix an IFS* $\mathcal{S} = \{S_i(x) = x/L + t_i\}_{i=1}^{M}$ *with random attractor* $\Lambda$ *and expectation matrices* $\mathcal{M} = \{\mathbf{M}_0, \ldots, \mathbf{M}_{L-1}\}$*. Assume that* $\mathcal{M}$ *is jointly positively irreducible. The Hausdorff, box and packing dimension of the random attractor* $\Lambda$ *have common value*

$$\dim \Lambda = \inf_{t \in [0,1]} \frac{P(t)}{\log L}, \tag{3.4}$$

*almost surely conditioned on non-extinction.*

### Applications

It is difficult to calculate or estimate this dimension formula numerically. We believe, when compared to the case in which there are either exact or negligible overlaps, no new phenomena can occur. One can make certain conclusions regarding the dimension of sets, such as:

- Assume that we are given a two-dimensional carpet with the property that the reciprocal of the contraction ratio ($L$) does not divide the number of maps ($M$). Consider a coin tossing system running on the cylinders of this IFS with parameter $p$. Fix any rational orthogonal projection of this IFS to a line. Then there exists a non-trivial parameter interval $(p_0, p_1)$ so that

  $$\dim_{\mathrm{H}}(\mathrm{proj}(\Lambda_p)) < \dim_{\mathrm{H}}(\Lambda_p).$$

- For coin tossing systems, there is no parameter interval of $p$ such that $\mathrm{Leb}(\Lambda_p) = 0$ but $\dim_{\mathrm{H}}(\Lambda) \geqslant 1$.

# 4 Examples

## 4.1 Projections of the right-angled Sierpiński carpet

In this section we consider projections of the right-angled Sierpiński carpet, the attractor of the following self-similar IFS in $\mathbb{R}^2$.

$$\mathcal{S} := \left\{ S_i(\underline{x}) = \frac{1}{2}\underline{x} + \underline{t}_i \right\}_{i=0}^{3},$$

where $\{\underline{t}_i\}_{i=0}^{3}$ is an enumeration of the set $\left\{0, \frac{1}{2}\right\}^2 \setminus \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$. For the early level approximations see Figure 3. We consider the coin tossing randomization on the deterministic IFS, with parameter $p \in (0, 1]$. The random attractor is

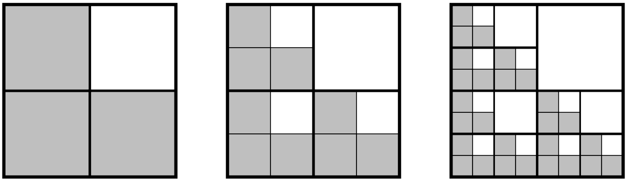

Figure 3: The approximations of the (deterministic) right-angled Sierpiński carpet.

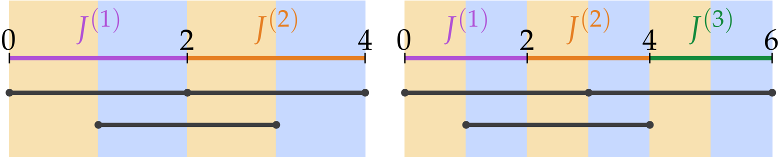

Figure 4: Depiction of the projected IFSs.

denoted by $\mathcal{G}_p$. The right-angled Sierpiński carpet $\mathcal{G}_p$ as well as its projections are non-empty if and only if $p > 1/3$, and its dimension is almost surely $\log_2(3p)$ (conditioned on non-extinction).

**Example 4.1** (45-degree projection of the random right-angled Sierpiński carpet)**.** Consider $\text{proj} : \mathbb{R}^2 \to \mathbb{R}$, $\text{proj}(x, y) := -x + y$, the rescaled version of the 45-degree projection. Then $\text{proj}(\mathcal{G}_p)$ (see Figure 3) gives the IFS $\mathcal{S} = \left\{S_i(x) = \frac{1}{2}x + 2(i-1)\right\}_{i=1}^{3}$. In this case $L = N = 2$. The types are determined by the basic intervals, $J^{(0)} := [0, 2]$ and $J^{(1)} := [2, 4]$. The environments are identified with $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots) \in \{0, 1\}^{\mathbb{N}}$.

The corresponding matrices are:

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Here we collect the applications of the theorems of this thesis.

- It follows from Application 3.3 that for $p \leqslant 0.672$ the Lebesgue measure of the random attractor is almost surely 0, however for $p \geqslant 0.673$ the Lebesgue measure is positive almost surely conditioned on non-extinction.

- It is clear that for $p = 1$ the random attractor (which agrees almost surely with the deterministic attractor) has non-empty interior. Since both matrices have spectral radius 1 it follows that for $p < 1$ the interior is empty almost surely. We mention that the same situation occurs if one considers the coordinate projections.

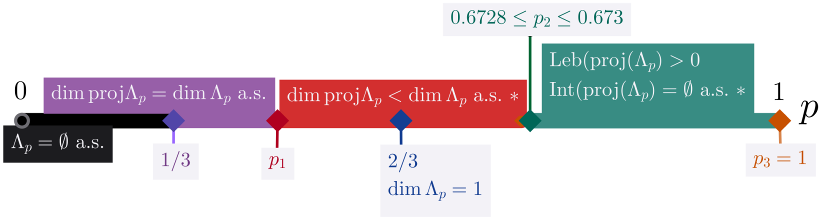

Figure 5: the parameter intervals in case of Example 4.1.

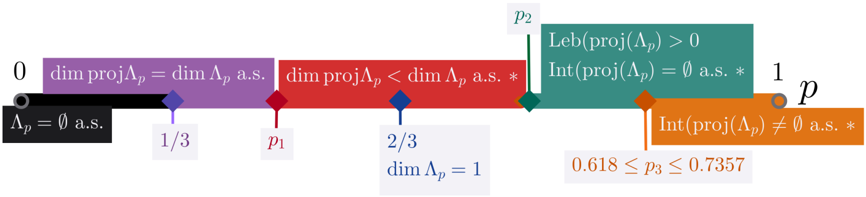

Figure 6: the parameter intervals in case of Example 4.2.

- It follows that there exists a parameter interval, where the interior is empty but the Lebesgue measure is positive almost surely conditioned on non-extinction.

This is summarized in Figure 5 as well as some corollaries of the dimension formula which we don't make explicit now.

**Example 4.2** (0-1-3, with contraction 1/2)**.**

$$S_i(x) = \frac{1}{2}x + t_i,\ t_i \in \{0, 1, 3\} \tag{4.1}$$

$J^{(0)} = [0, 2]$, $J^{(1)} = [2, 4]$, $J^{(2)} = [4, 6]$.

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

In Application 3.14 we proved that for $p > 0.7357$ the interior of the random attractor is not empty almost surely on non-extinction. This behaviour is different from the 45-degree and the axis projections. We can apply Lemma 3.8 to show that the interesting parameter interval exists. This, together with some corollaries of the dimension formula is summarized in Figure 6.

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