POSITIVITY OF LEBESGUE MEASURE OF SOME STATISTICALLY SELF-SIMILAR SETS

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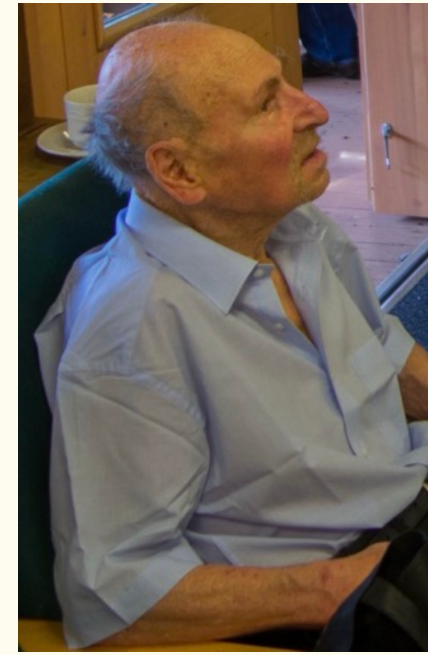
2026 April 16

based on a joint work with Károly Simon

PROJECTIONS

MARSTRAND'S PROJECTION THEOREM

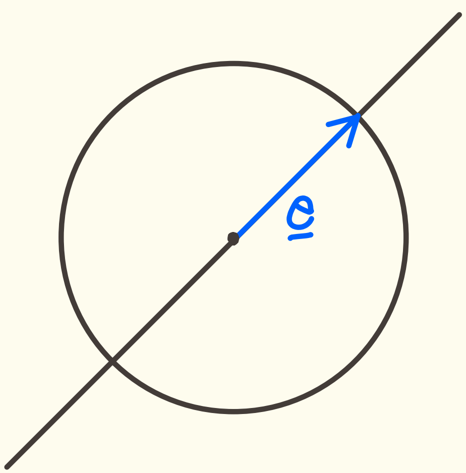


$E \subseteq \mathbb{R}^2$, BOREL SET (1954)

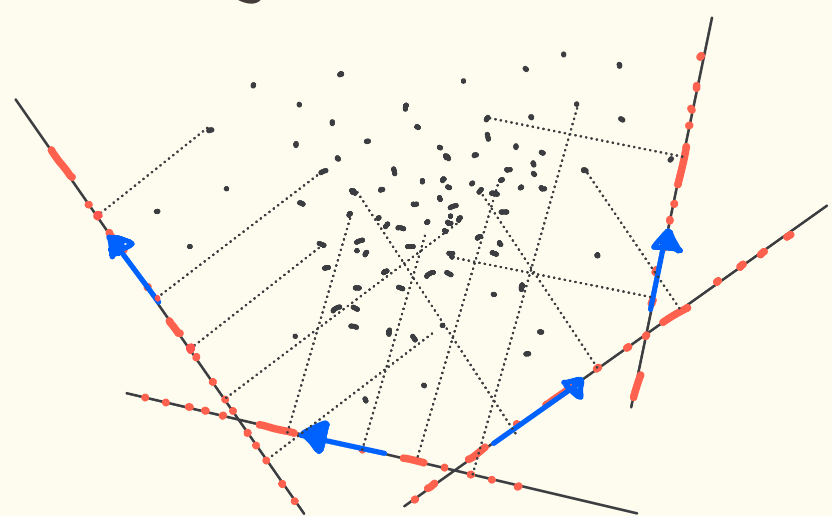
1. If $\dim_H E > 1$ then for almost every $\underline{e} \in S^1$ $\mathrm{Leb}(\mathrm{proj}_{\underline{e}} E) > 0$.

2. If $\dim_H E \leq 1$ then for almost every $\underline{e} \in S^1$

$$\dim_H(\mathrm{proj}_{\underline{e}} E) = \dim_H E.$$



$\mathrm{proj}_{\underline{e}}$ = orth. projection to line spanned by $\underline{e}$



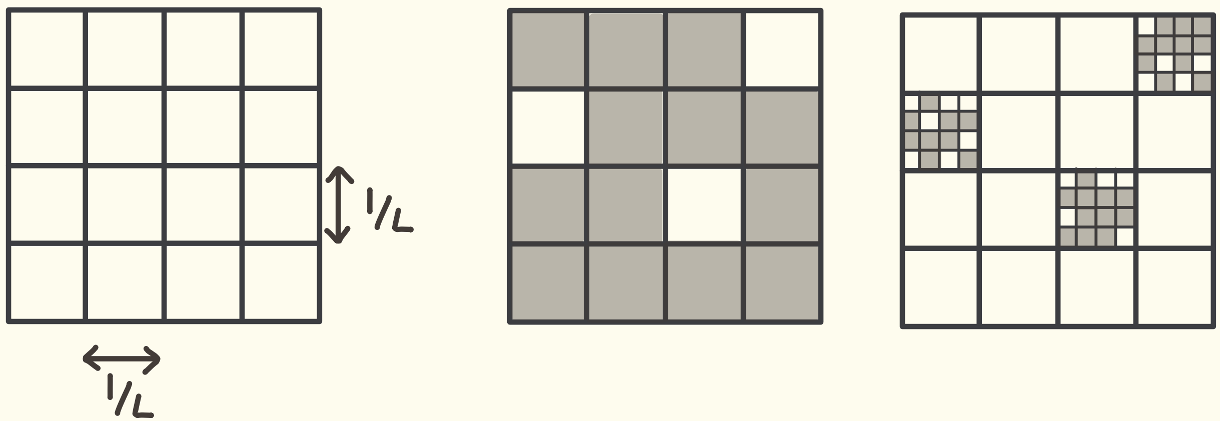
Mandelbrot Percolation

$E \subseteq \mathbb{R}^2$, Borel set

① If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.

② If $\dim_H E \le 1$ then for almost every $e \in S^1$ $\dim_H(\mathrm{proj}_e E) = \dim_H E$.

A/ Construction: Fix $p \in (0,1]$, $L \ge 2$.



$\dots \quad M_p$

$$\mathbb{P}(M_p \neq \emptyset) > 0 \iff p > 1/L^2$$

MANDELBROT PERCOLATION

$E \subseteq \mathbb{R}^2$, BOREL SET
① If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.
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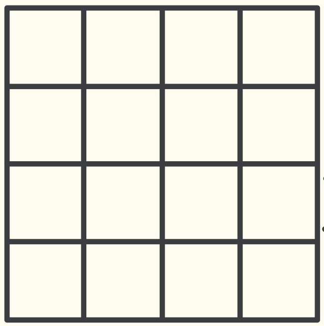
A/ Construction: Fix $p \in (0,1]$, $L \geq 2$.



$1/L$

$1/L$

- $\mathbb{P}(M_p \neq \emptyset) > 0 \iff p > 1/L^2$
- $\dim M_p = \dfrac{\log L^2 p}{\log L}$ a.s. conditioned on non-extinction

[Hawkes '81
Falconer '86
Mauldin & Williams '86
Kahane '85]

a.s.* = a.s. conditioned on non-extinction

B/ Rams-Simon, '14, '15

① $\dim_H M_p > 1 \implies \mathrm{Int}(\mathrm{proj}_e M_p) \neq \emptyset$

② $\dim_H M_p \le 1 \implies \dim_H(\mathrm{proj}_e M_p) = \dim_H M_p$

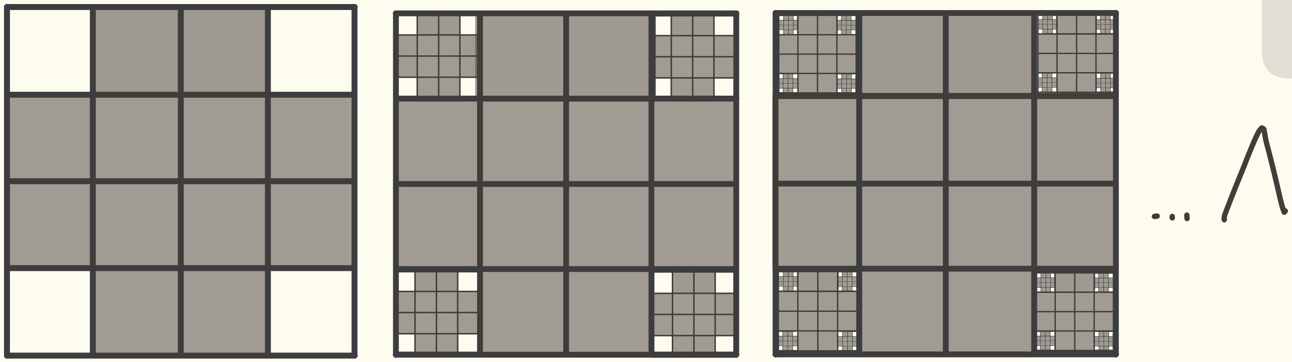
} a.s.* for all proj_e projections to lines.

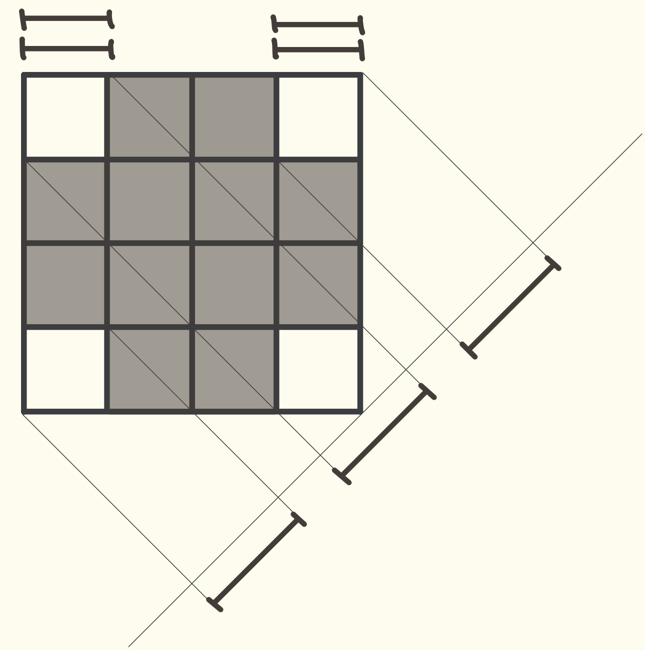
Exceptional Directions:

$E \subseteq \mathbb{R}^2$, BOREL SET

① If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.

② If $\dim_H E \leq 1$ then for almost every $e \in S$ $\dim_H(\mathrm{proj}_e E) = \dim_H E$.

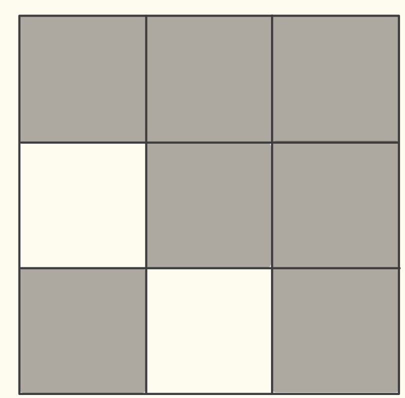




many rational directions are exceptional:

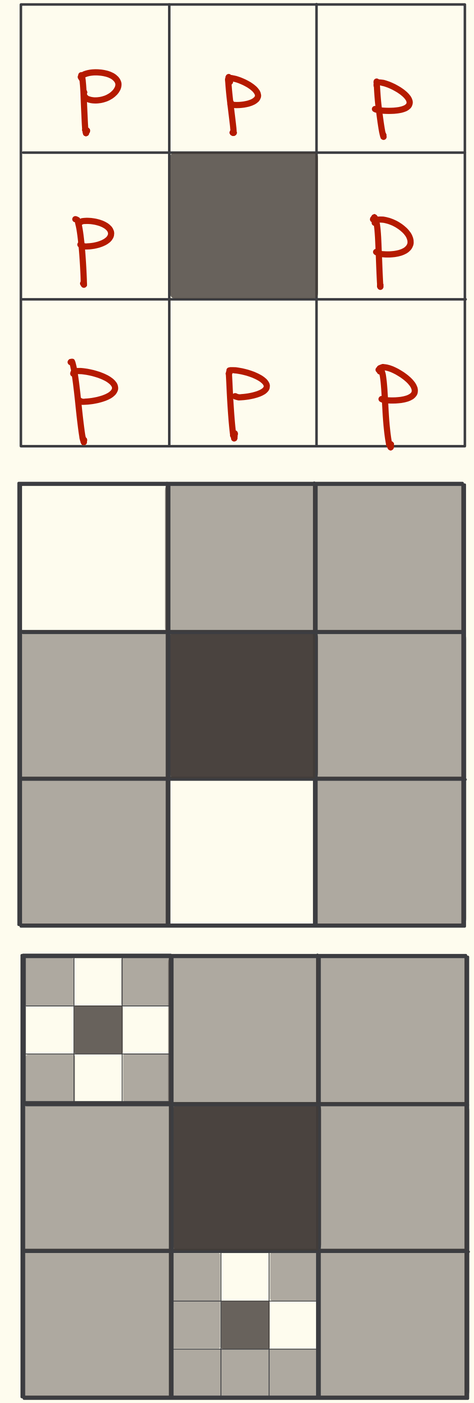
$$\dim(\mathrm{proj}\,\Lambda) < \dim(\Lambda) = 1$$

Generally: $\mathbb{Z}$-grid aligned sets
+ rational projections
can be problematic



EXCEPTIONAL DIRECTIONS: Randomness

$p \in (0,1]$

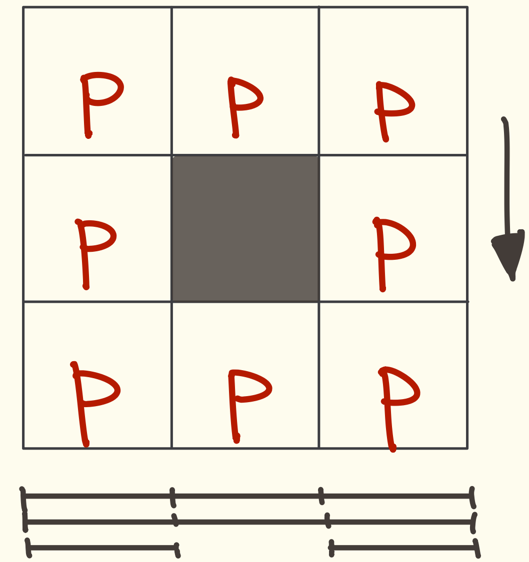


$\dots \Lambda_p$

EXCEPTIONAL DIRECTIONS: Randomness

$p \in (0,1]$

Λ_p

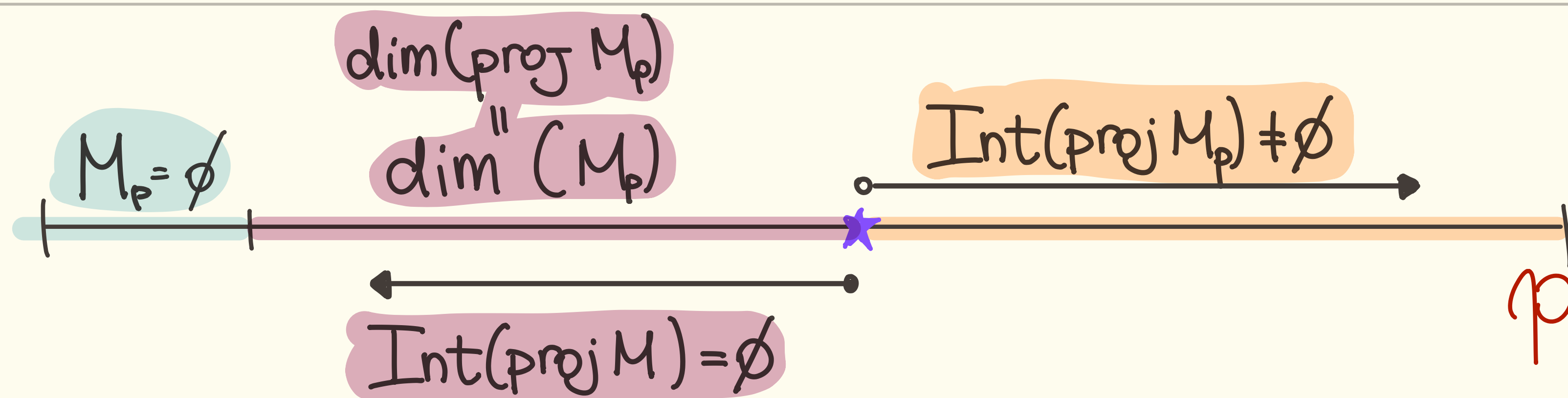


$\text{proj}\,\Lambda_p$

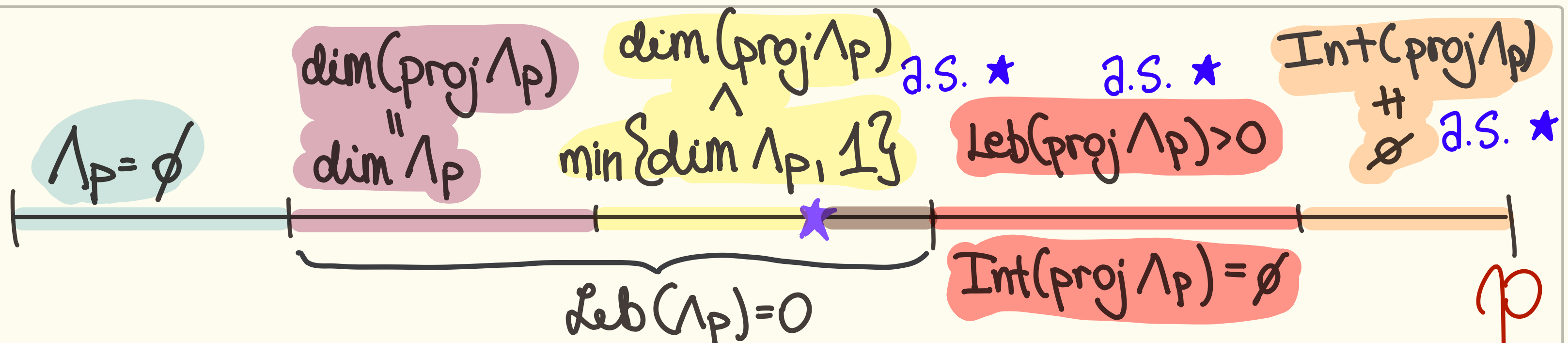
(1) orthogonal projections to the x-axis:

- Dekking-Grimmett '88
- Falconer '89
- Falconer-Grimmet '92

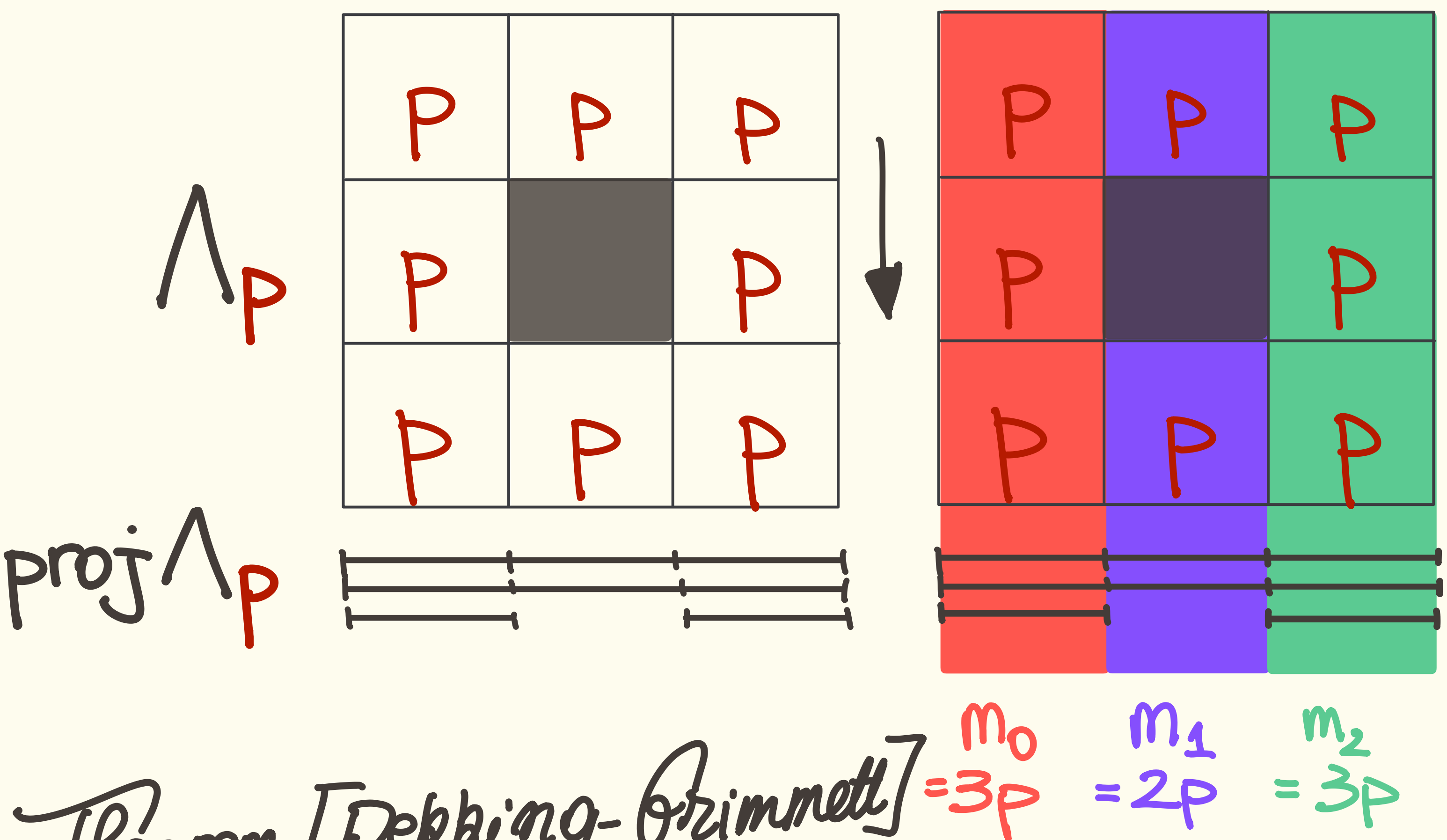
projections of the Mandelbrot percolation:



$\text{proj}\,\Lambda_p$:



POSITIVITY OF LEBESGUE MEASURE

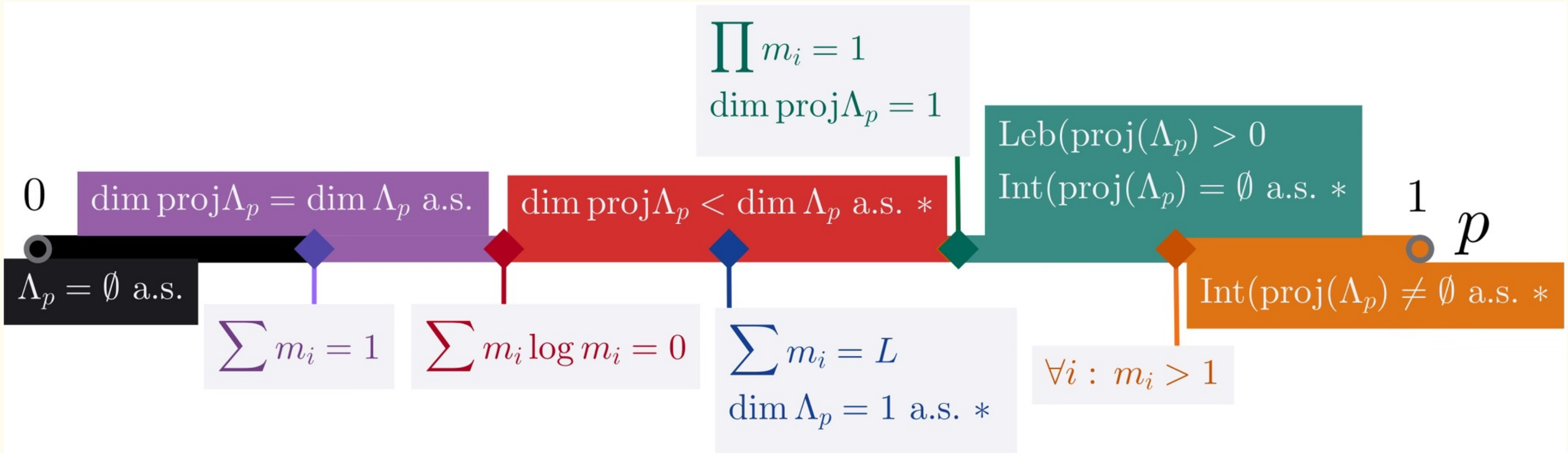


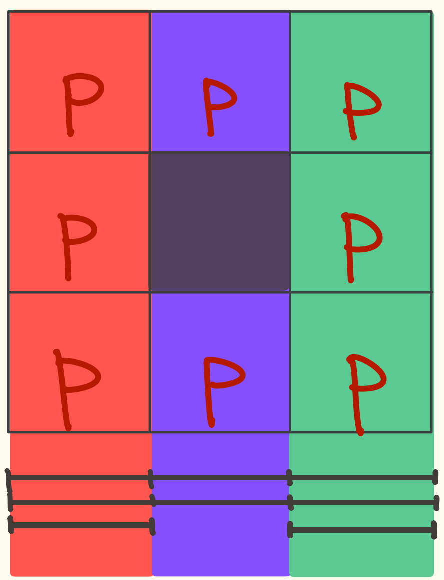
$$m = \prod_{i=0}^{L-1} m_i = m_0 m_1 m_2$$

Theorem [Dekking-Grimmett]

$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$ a.s. conditioned on non-extinction $\iff m = m_0 m_1 m_2 > 1$

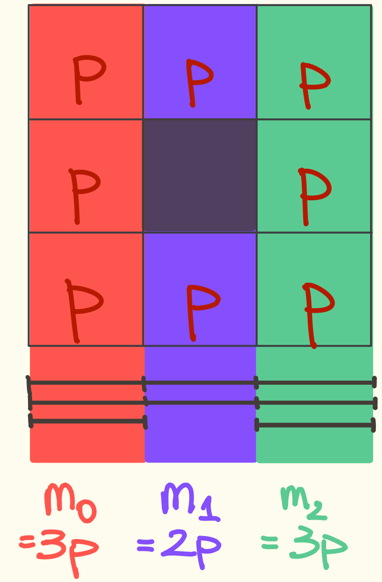
PARAMETER INTERVALS





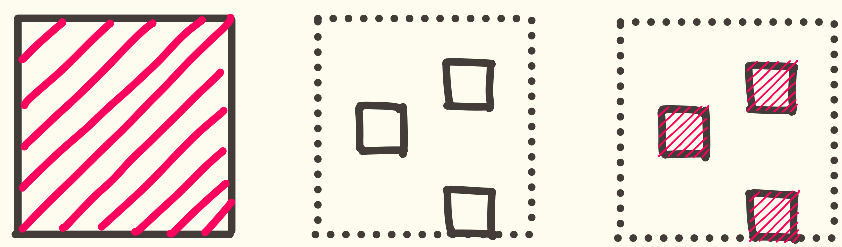
$m_0 = 3p$ $m_1 = 2p$ $m_2 = 3p$

$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$
a.s. conditioned on non-extinction $\Leftrightarrow m > 0$



$$m = \prod_{i=0}^{L-1} m_i = m_0 m_1 m_2$$

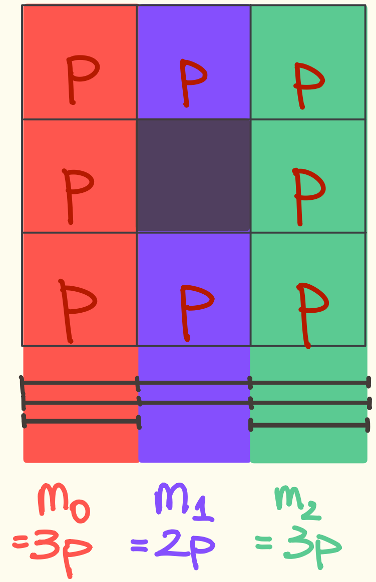
$\boxed{1.}$ $\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$
a.s. conditioned on non-extinction $\Leftrightarrow \mathbb{P}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0) > 0$



$\boxed{2.}$ $\mathbb{P}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0) > 0 \Leftrightarrow \mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) > 0$

$\boxed{3.}$ $\mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) > 0 \Leftrightarrow \lim_{n\to\infty} \mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p^n)) > 0$

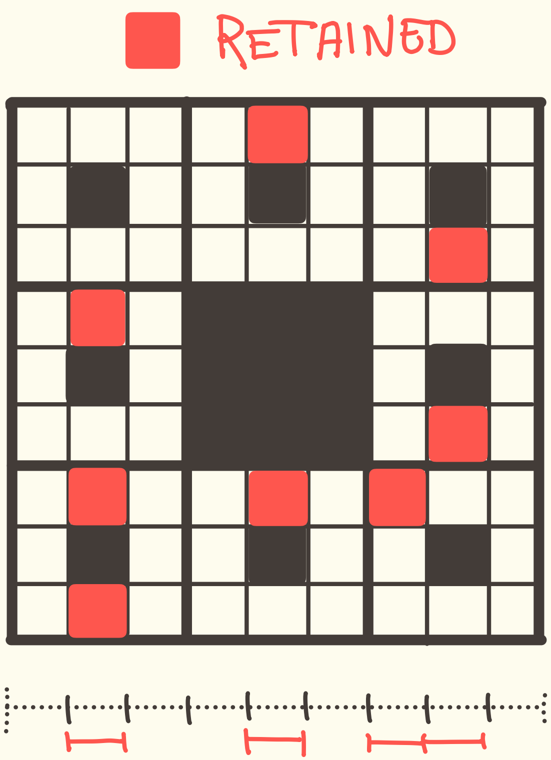
$$\lim_{n\to\infty} \mathbb{E}\left(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p^n)\right) > 0 \iff m > 0$$



$$m = \prod_{i=0}^{L-1} m_i = m_0 m_1 m_2$$

$$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p^n) = 3^{-n} \cdot \# Y_n$$

$$\mathbb{E}\left(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p^n)\right) = 3^{-n} \cdot \sum_{\underline{i}\in[3]^n} \mathbb{P}(\underline{i} \in Y_n)$$



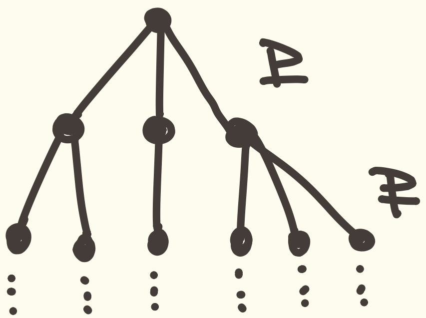
$\{\underline{i} \in \{0,1,2\}^n :$ the triadic interval corresponding to $\underline{i}$ is retained$\}$

$\parallel$

Y_n

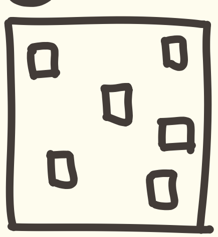
KEY RELATIONSHIP 1. BRANCHING PROCESSES

G-W process
P offspring distr.



Z_n = # level n nodes

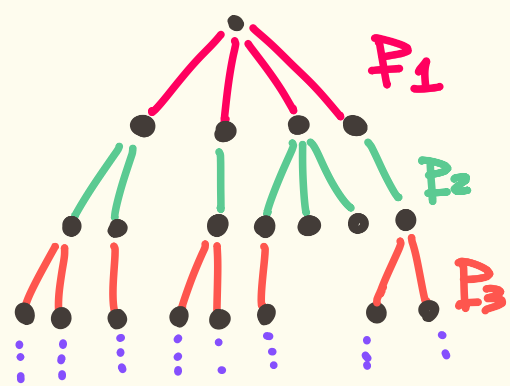
e.g.



retained level-n squares

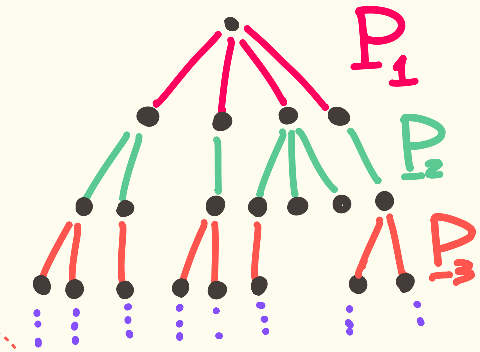
branching processes in varying environments

$(P_1, P_2, P_3, P_4, \ldots)$



branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \ldots)$ is a random sequence of distributions.

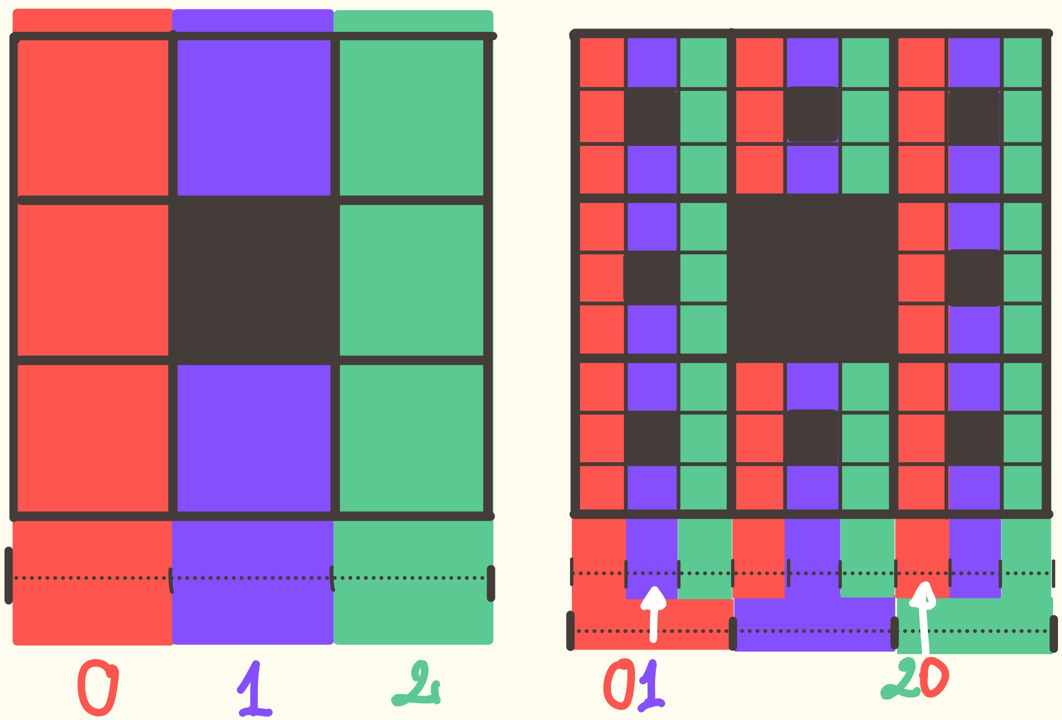


e.g. (q_1, q_2, q_3) distr.

$\quad\ \ \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}$

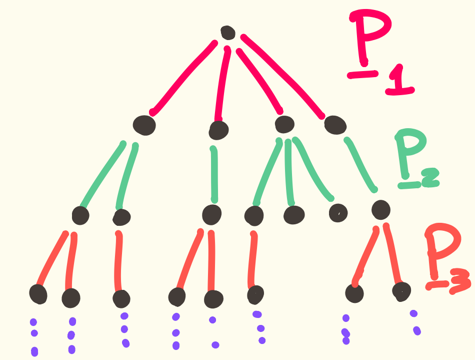
$$\left(\underset{1.}{\cdot}, \underset{2.}{\cdot}, \underset{3.}{\cdot}, \underset{4.}{\cdot}, \underset{\ldots}{\ldots} \right)$$

KEY RELATIONSHIP 1. BPREs



branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \dots)$ is a random seq. of distr.



$(i_1, \dots, i_n) = \underline{i} \in \{0,1,2\}^n$

$(\mathrm{Bin}(3,p), \mathrm{Bin}(2,p), \mathrm{Bin}(3p))$

q_0 q_1 q_2

$\mathbb{P}(q_0) = 1/3$ $\mathbb{P}(q_1) = 1/3$ $\mathbb{P}(q_2) = 1/3$

$\tilde{Z}_n(\underline{i}) = \#\{$num of level-n children given $q_{i_1}, \dots, q_{i_n}\}$

$\tilde{Z}_n = \#\{$num of level-n children$\}$

$$3^{-n} \sum_{\underline{i} \in [3]^n} \mathbb{P}(i \in Y_n) = \frac{1}{3^n} \sum_{\underline{i} \in [3]^n} \mathbb{P}(\tilde{Z}_n(\underline{i}) > 0) = \mathbb{P}(\tilde{Z}_n > 0)$$

g

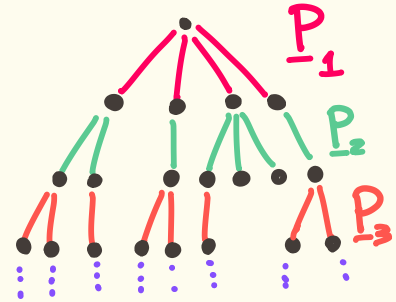
SURVIVAL OF BPREs

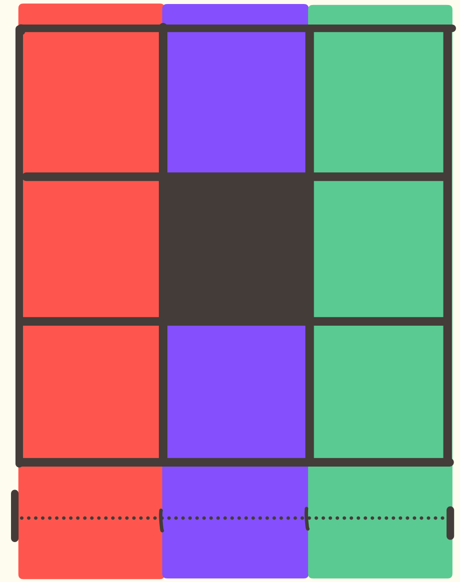
$\tilde{Z}_n$ is a branching process in a random environment.

QUESTION: $\lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0)$?

branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \dots)$ is a random seq. of distr.





THEOREM [ATHREYA-KARLIN]:

$$\lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0) > 0$$

$$\Updownarrow$$

$$m = \mathbb{E}(P_0)\cdot \mathbb{E}(P_1)\cdot \mathbb{E}(P_2) > 1$$

$\tilde{Z}_n$ b.p.r.e.

$(\mathrm{Bin}(3,p),\ \mathrm{Bin}(2,p),\ \mathrm{Bin}(3,p))$

$P_0 \quad P_1 \quad P_2$

$\mathbb{P}(P_0) = 1/3 \quad \mathbb{P}(P_1) = 1/3 \quad \mathbb{P}(P_2) = 1/3$

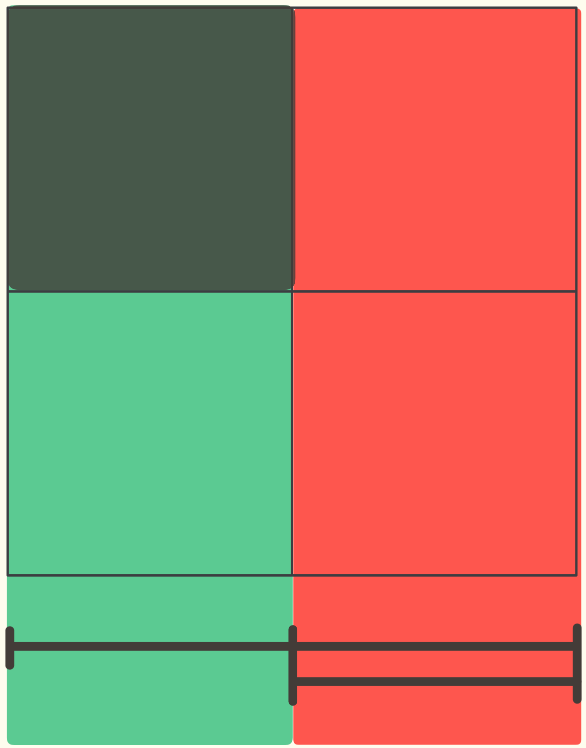
$\mathbb{E}(P_0) = 3p \quad \mathbb{E}(P_1) = 2p \quad \mathbb{E}(P_2) = 3p$

Summary:

$$\mathbb{P}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0) > 0 \iff \mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) > 0$$

$$\mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) = \mathbb{P}(\tilde{Z}_n > 0)$$

RATIONAL PROJ.

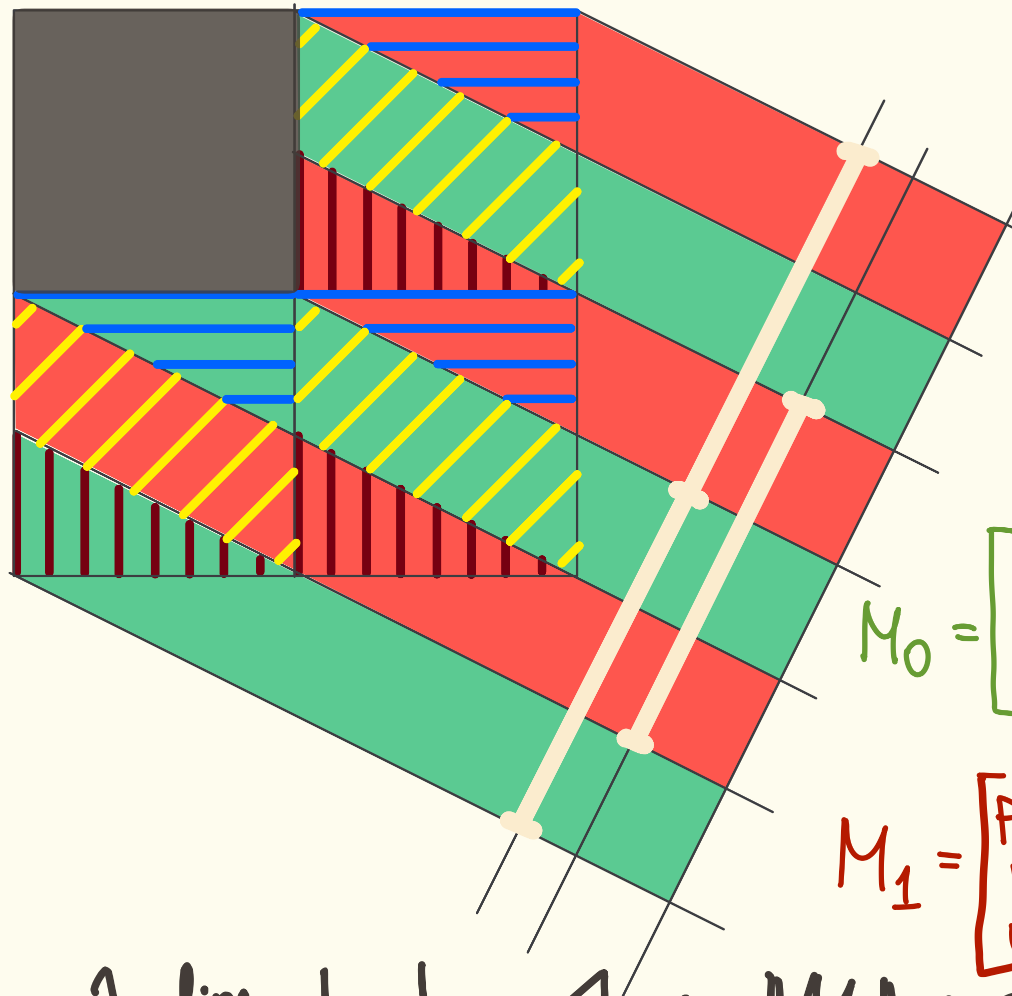


$m_0 = p$, $m_1 = 2p$

$\log m_0 \cdot m_1 > 0 \iff$

$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$

branching process in a random environment $\to$ multitype bpre



$M_0 = \begin{bmatrix} p & 0 & 0 \\ 0 & p & p \\ 0 & p & 0 \end{bmatrix}$,

$M_1 = \begin{bmatrix} p & p & 0 \\ p & 0 & p \\ 0 & 0 & p \end{bmatrix}$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{L^n} \cdot \sum_{\underline{i} \in [L]^n} \log \| M_{\underline{i}} \| > 0$$

given:

a/ all matrices M_i has a pos. element in all rows & cols.

b/ there is a strictly positive product $M_{i_1} M_{i_2} \cdots M_{i_k}$

MBPRE THM [O.-Simon, 2026]

$(\tilde{Z}_n)_n$ is an MBPRE with

a/ all matrices M_i has a pos. element in all rows & cols.

b/ there is a strictly positive product $M_{i_1} M_{i_2} \dots M_{i_k}$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{L^n} \cdot \sum_{\underline{i} \in [L]^n} \log \|M_{\underline{i}}\| > 0 \iff \lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0) > 0$$

COROLLARY. [O.-Simon, 2026]

$S = \{S_i(x) = \frac{x}{L} + t_i\}$, $L \in \mathbb{N}$, $t_i \in \mathbb{R}$.

$M_0, \dots, M_{L-1}$ are the corresponding expectation matrices.

under red conditions on $M_0, \dots, M_{L-1}$:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{L^n} \cdot \sum_{\underline{i} \in [L]^n} \log \|M_{\underline{i}}\| > 0 \iff \mathcal{L}eb(\Lambda_{S,p}) > 0$$

a.s. *

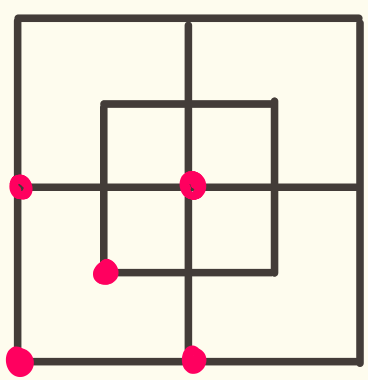
OPEN QUESTIONS.

1) do we need:

a/ all matrices M_i has a pos. element in all rows & cols.,

b/ there is a strictly positive product $M_{i_1} M_{i_2} \dots M_{i_k}$?

Remark. Generally holds for 1-dimensional systems, but not for



(not for this system for example.)

MBPRE THM [O.-Simon, 2026]

$(Z_n)_n$ is an MBPRE with

a/ all matrices M_i has a pos. element in all rows & cols.

b/ there is a strictly positive product $M_{i_1} M_{i_2} \dots M_{i_k}$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{L^n} \cdot \sum_{\underline{i} \in [L]^n} \log \| M_{\underline{i}} \| > 0 \iff \lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0) > 0$$

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$S = \{ S_i(x) = \frac{x}{L} + t_i \}$, $L \in \mathbb{N}$, $t_i \in \mathbb{R}$.

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a.s. *

OPEN QUESTIONS.

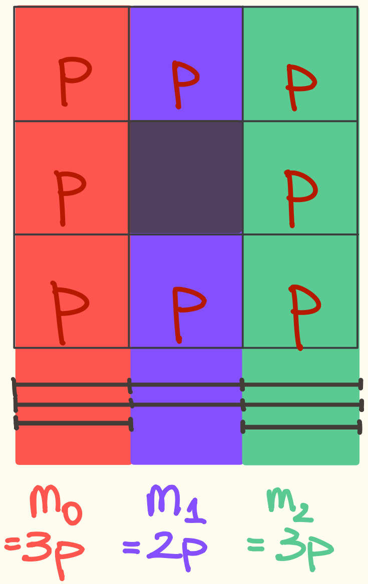
2) Can we broaden this family?

$$S = \{ S_i(x) = 1/2 + t_i \},\ i \in \mathbb{N},\ t_i \in \mathbb{R}.$$

THANK YOU

FOR YOUR ATTENTION!

EMPTY INTERIOR



Theorem [Falconer-Grimmett]:

if $\exists i$ $m_i < 1 \Rightarrow \mathrm{Int}(\mathrm{proj}\,\Lambda_p) = \phi$ a.s.

„bad" behavior repeats inside all column.