POSITIVITY OF LEBESGUE MEASURE OF SOME STATISTICALLY SELF-SIMILAR SETS

ORGOVÁNYI VILMA

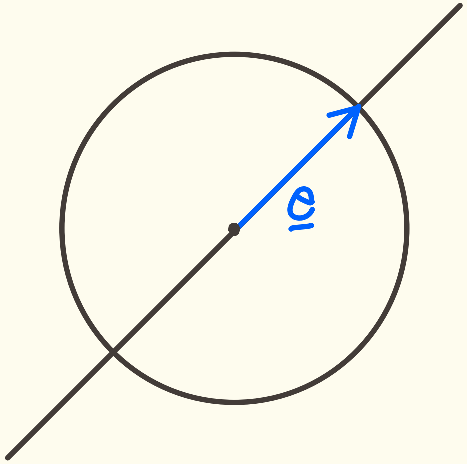
2026 April 16

based on a joint work with Károly Simon

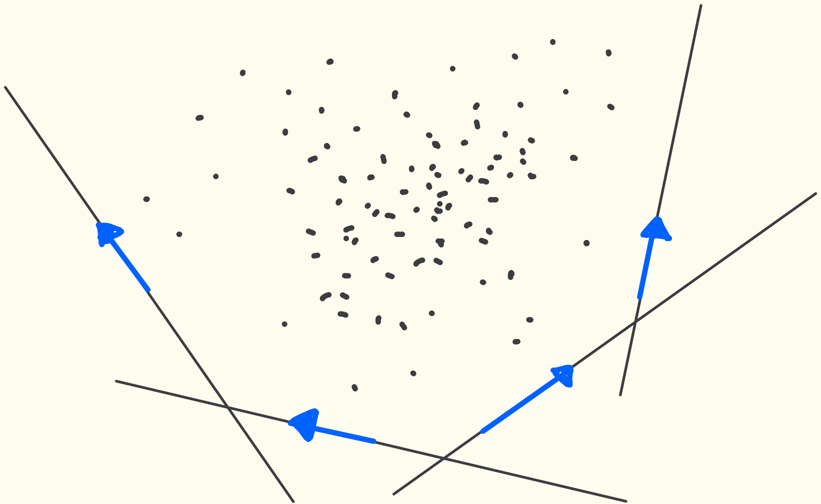
PROJECTIONS

MARSTRAND'S PROJECTION THEOREM

$E \subseteq \mathbb{R}^2$, BOREL SET (1954)



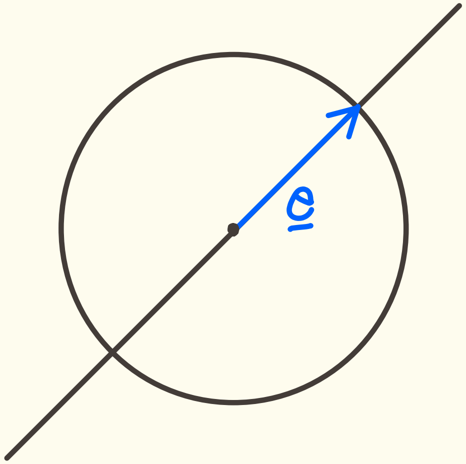
$\mathrm{proj}_{\underline{e}}$ = orth. projection to line spanned by $\underline{e}$



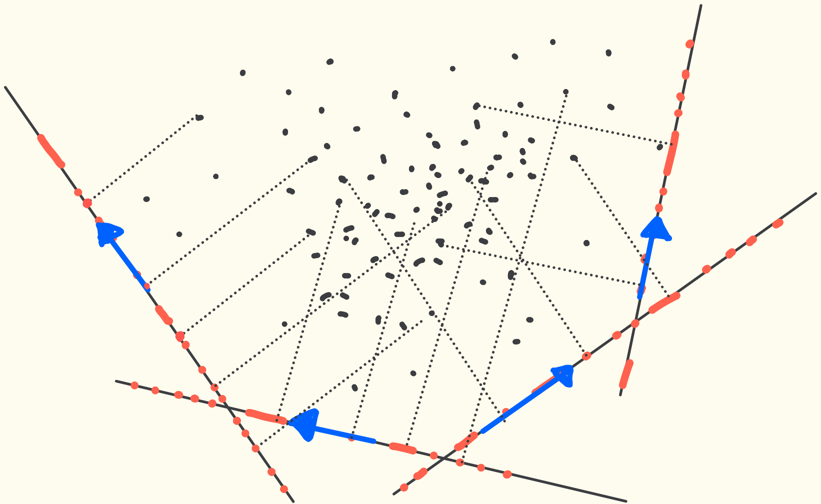
PROJECTIONS

MARSTRAND'S PROJECTION THEOREM

$E \subseteq \mathbb{R}^2$, BOREL SET (1954)



proj_{e} = orth. projection to line spanned by e



PROJECTIONS

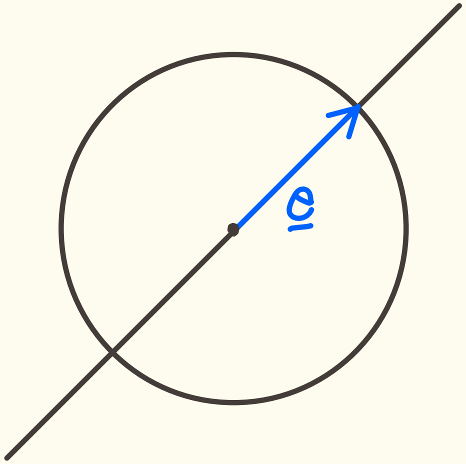
MARSTRAND'S PROJECTION THEOREM

$E \subseteq \mathbb{R}^2$, BOREL SET (1954)

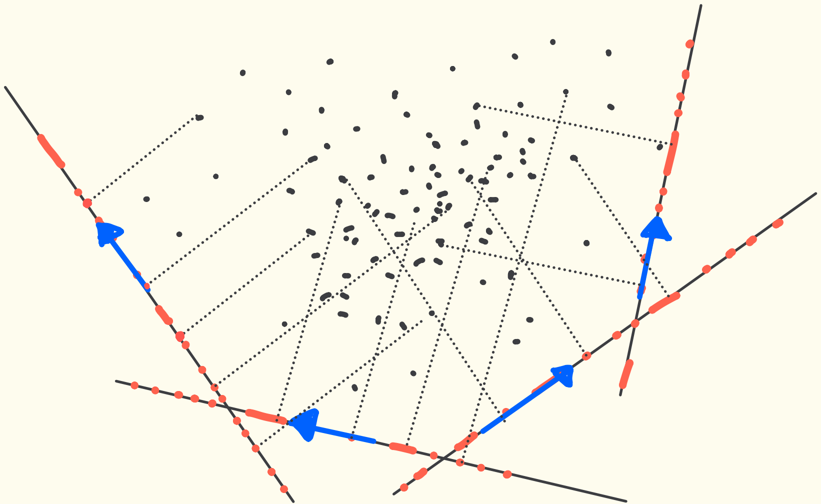
① If $\dim_H E > 1$ then for almost every $\underline{e} \in S^1$ $\mathrm{Leb}(\mathrm{proj}_{\underline{e}} E) > 0$.

② If $\dim_H E \leq 1$ then for almost every $\underline{e} \in S^1$

$$\dim_H(\mathrm{proj}_{\underline{e}} E) = \dim_H E.$$



$\mathrm{proj}_{\underline{e}}$ = orth. projection to line spanned by $\underline{e}$



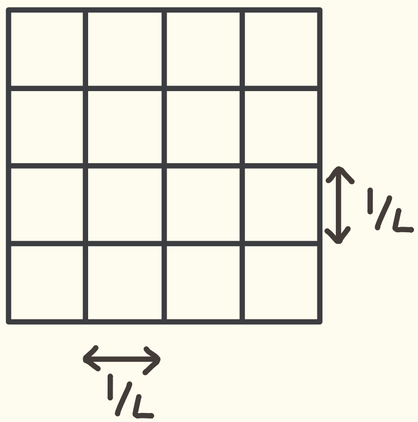
MANDELBROT PERCOLATION

$E \subseteq \mathbb{R}^2$, BOREL SET

① If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.

② If $\dim_H E \leq 1$ then for almost every $e \in S^1$ $\dim_H(\mathrm{proj}_e E) = \dim_H E$.

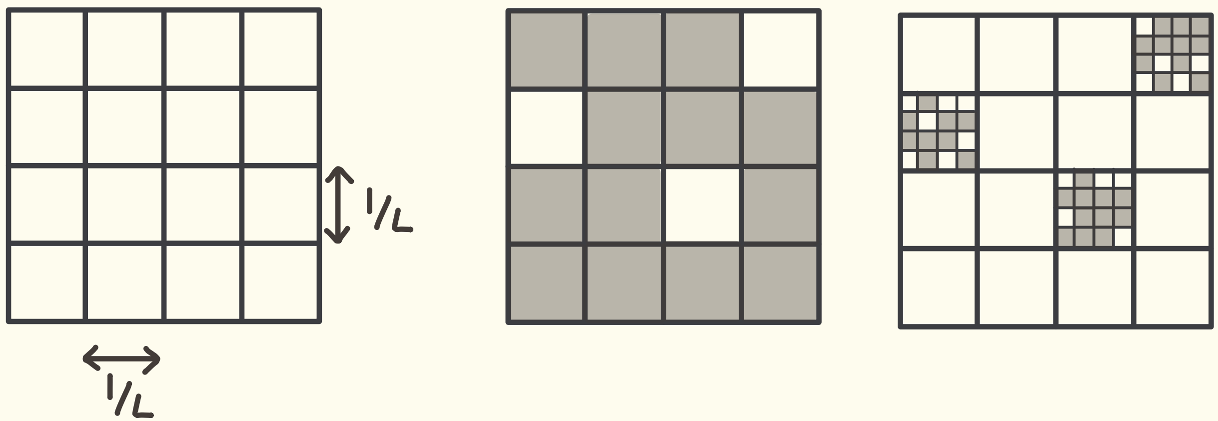
A/ Construction: Fix $p \in (0,1]$, $L \geq 2$.



MANDELBROT PERCOLATION

$E \subseteq \mathbb{R}^2$, BOREL SET
1. If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.
2. If $\dim_H E \leq 1$ then for almost every $e \in S^1$ $\dim_H(\mathrm{proj}_e E) = \dim_H E$.

A/ Construction: Fix $p \in (0,1]$, $L \geq 2$.



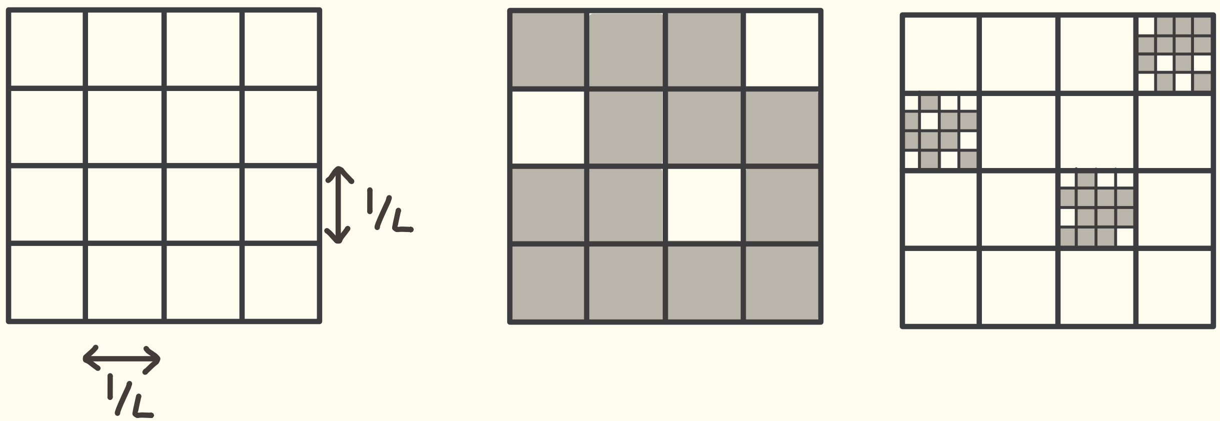
. . .

Mandelbrot Percolation

$E \subseteq \mathbb{R}^2$, BOREL SET

1. If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.
2. If $\dim_H E \leq 1$ then for almost every $e \in S^1$ $\dim_H(\mathrm{proj}_e E) = \dim_H E$.

A/ Construction: Fix $p \in (0,1]$, $L \geq 2$.



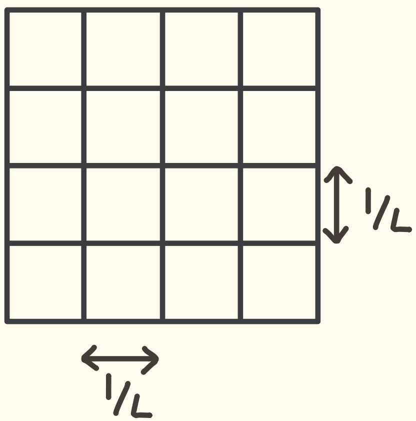
$\ldots \quad M_p$

$$\mathbb{P}(M_p \neq \emptyset) > 0 \iff p > 1/L^2$$

MANDELBROT PERCOLATION

$E \subseteq \mathbb{R}^2$, BOREL SET
① If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.
② If $\dim_H E \leq 1$ then for almost every $e \in S^1$ $\dim_H(\mathrm{proj}_e E) = \dim_H E$.

A/ Construction: Fix $p \in (0,1]$, $L \geq 2$.



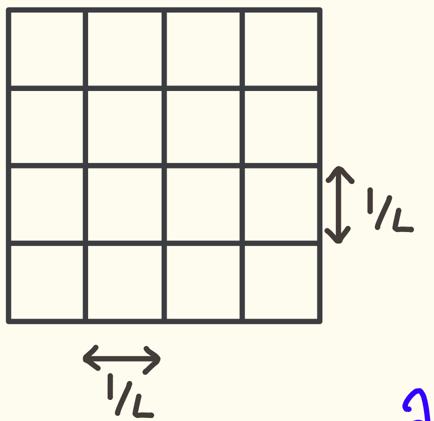
- $\mathbb{P}(M_p \neq \emptyset) > 0 \iff p > 1/L^2$
- $\dim M_p = \dfrac{\log L^2 p}{\log L}$ a.s. conditioned on non-extinction

$\left[\begin{array}{l} \text{Hawkes '81} \\ \text{Falconer '86} \\ \text{Mauldin \& Williams '86} \\ \text{Kahane '85} \end{array}\right]$

MANDELBROT PERCOLATION

$E \subseteq \mathbb{R}^2$, BOREL SET
① If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.
② If $\dim_H E \leq 1$ then for almost every $e \in S^1$ $\dim_H(\mathrm{proj}_e E) = \dim_H E$.

A/ Construction: Fix $p \in (0,1]$, $L \geq 2$.



- $\mathbb{P}(M_p \neq \emptyset) > 0 \iff p > 1/L^2$
- $\dim M_p = \dfrac{\log L^2 p}{\log L}$ a.s. conditioned on non-extinction

[Hawkes '81
Falconer '86
Mauldin & Williams '86
Kahane '85]

a.s.* = a.s. conditioned on non-extinction

B/ Rams-Simon, '14, '15

a.s. ★

① $\dim_H M_p > 1 \implies \mathrm{Int}(\mathrm{proj}_e M_p) \neq \emptyset$

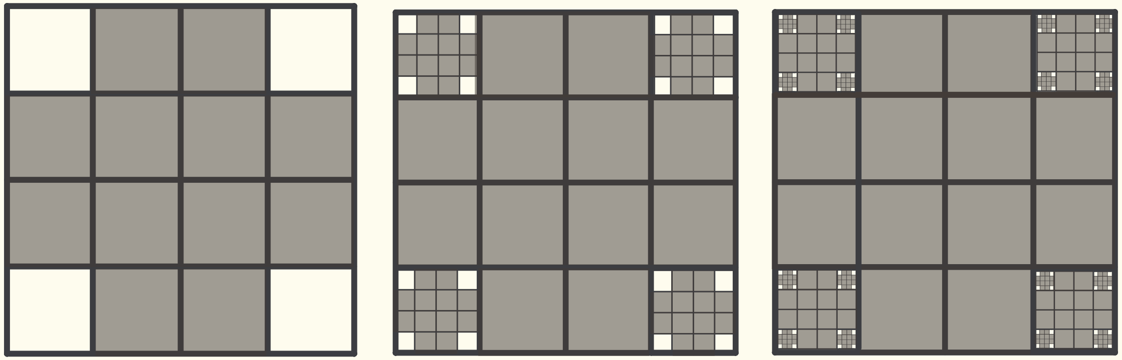
② $\dim_H M_p \leq 1 \implies \dim_H(\mathrm{proj}_e M_p) = \dim_H M_p$

for all proj_e projections to lines.

Exceptional Directions:

$E \subseteq \mathbb{R}^2$, BOREL SET

1. If $\dim_H E > 1$ then for almost every $\underline{e} \in S^1$ $\mathrm{Leb}(\mathrm{proj}_{\underline{e}} E) > 0$.
2. If $\dim_H E \leq 1$ then for almost every $\underline{e} \in S^1$ $\dim_H(\mathrm{proj}_{\underline{e}} E) = \dim_H E$.



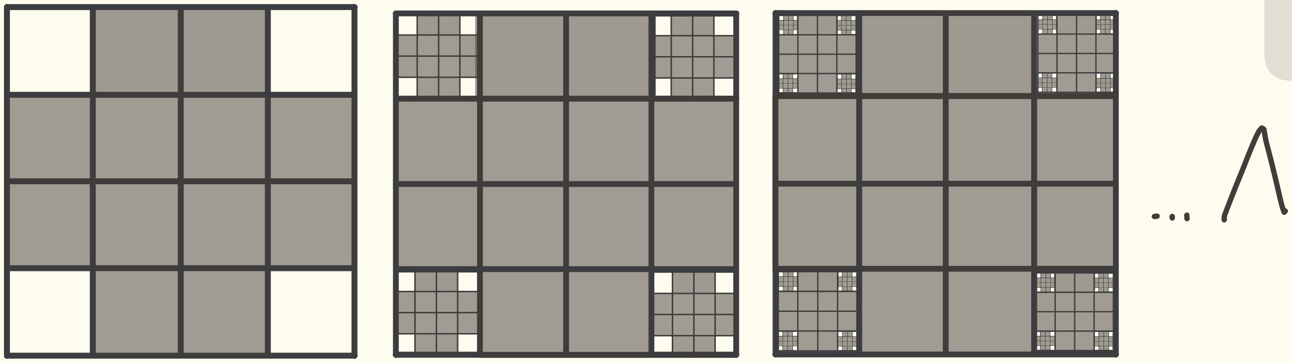
$\dots \quad \Lambda$

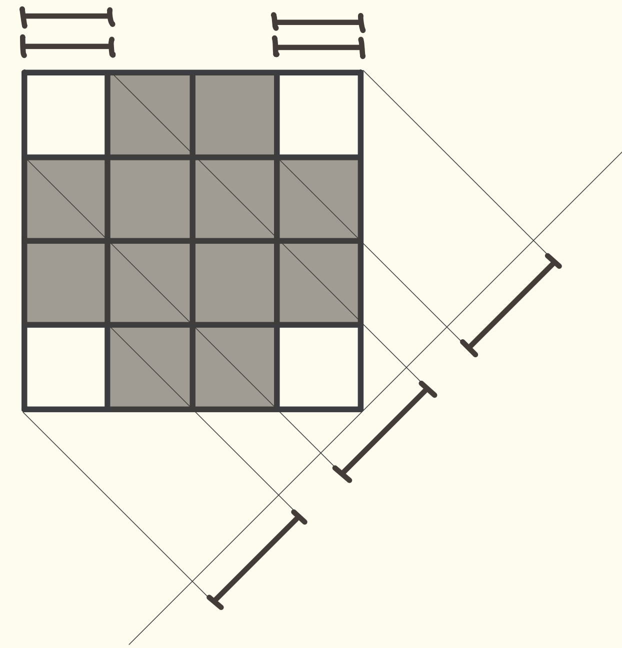
EXCEPTIONAL DIRECTIONS:

$E \subseteq \mathbb{R}^2$, BOREL SET

1. If $\dim_H E > 1$ then for almost every $\underline{e} \in S^1$ $\mathrm{Leb}(\mathrm{proj}_{\underline{e}} E) > 0$.

2. If $\dim_H E \leq 1$ then for almost every $\underline{e} \in S^1$ $\dim_H(\mathrm{proj}_{\underline{e}} E) = \dim_H E$.





many rational directions are exceptional:

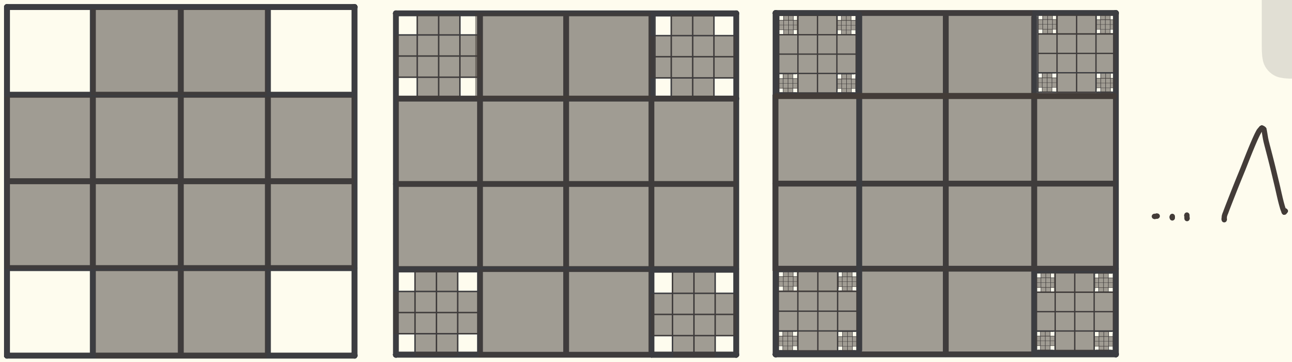
$$\dim(\mathrm{proj}\,\Lambda) < \dim(\Lambda) = 1$$

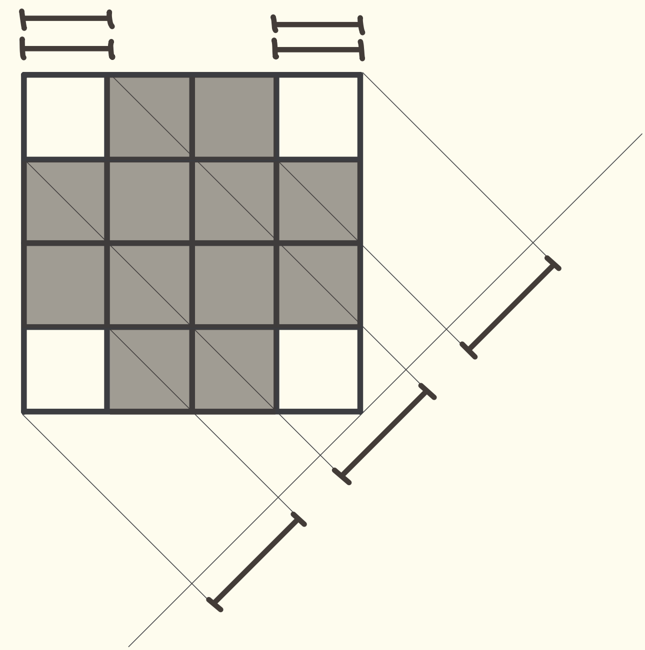
Exceptional Directions:

$E \subseteq \mathbb{R}^2$, Borel set

① If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.

② If $\dim_H E \le 1$ then for almost every $e \in S$ $\dim_H(\mathrm{proj}_e E) = \dim_H E$.

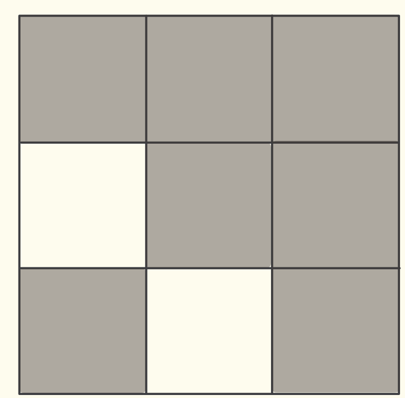




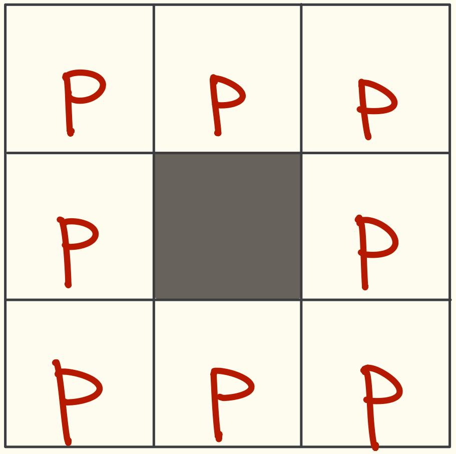
many rational directions are exceptional:

$$\dim(\mathrm{proj}\,\Lambda) < \dim(\Lambda) = 1$$

Generally: $\mathbb{Z}$-grid aligned sets
+ rational projections
can be problematic



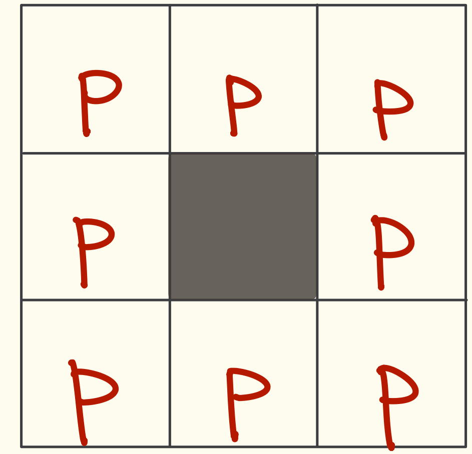
Exceptional Directions: Randomness

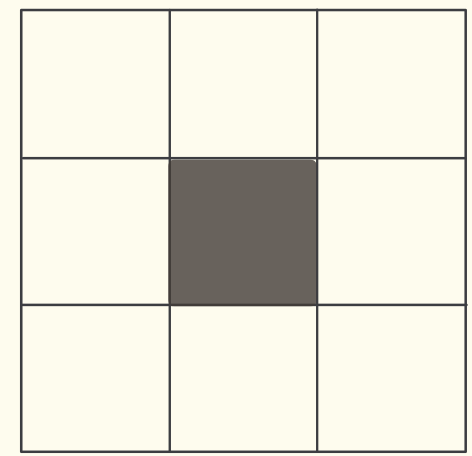


$p \in (0, 1]$

Exceptional Directions: Randomness

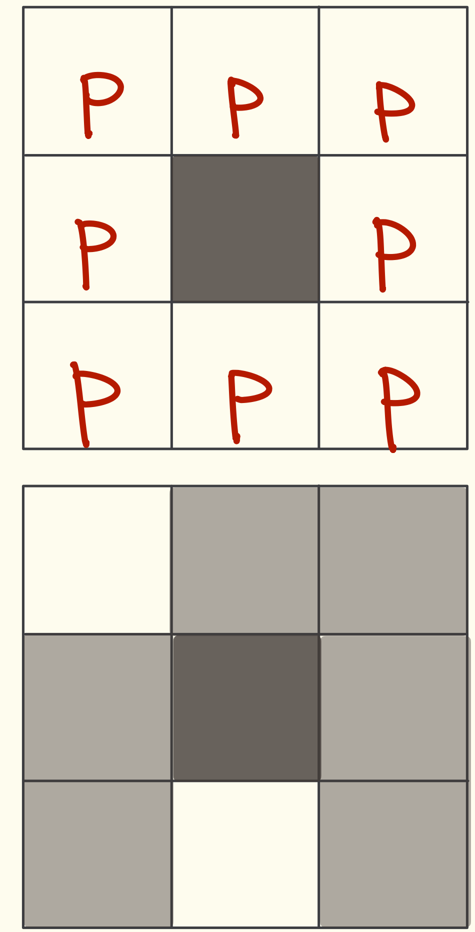
$p \in (0,1]$





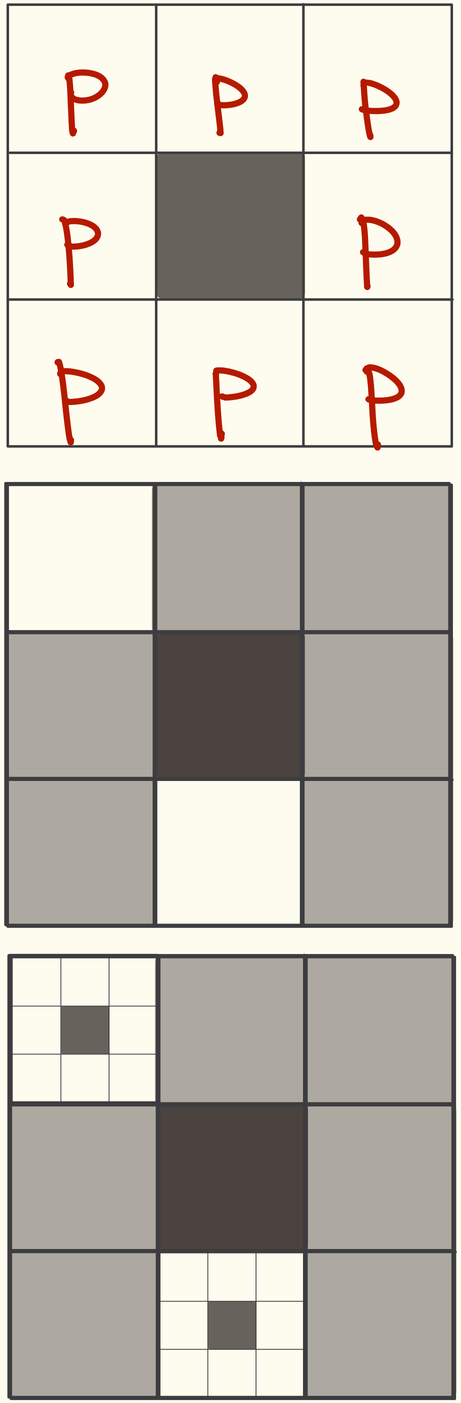
Exceptional Directions: Randomness

$p \in (0,1]$



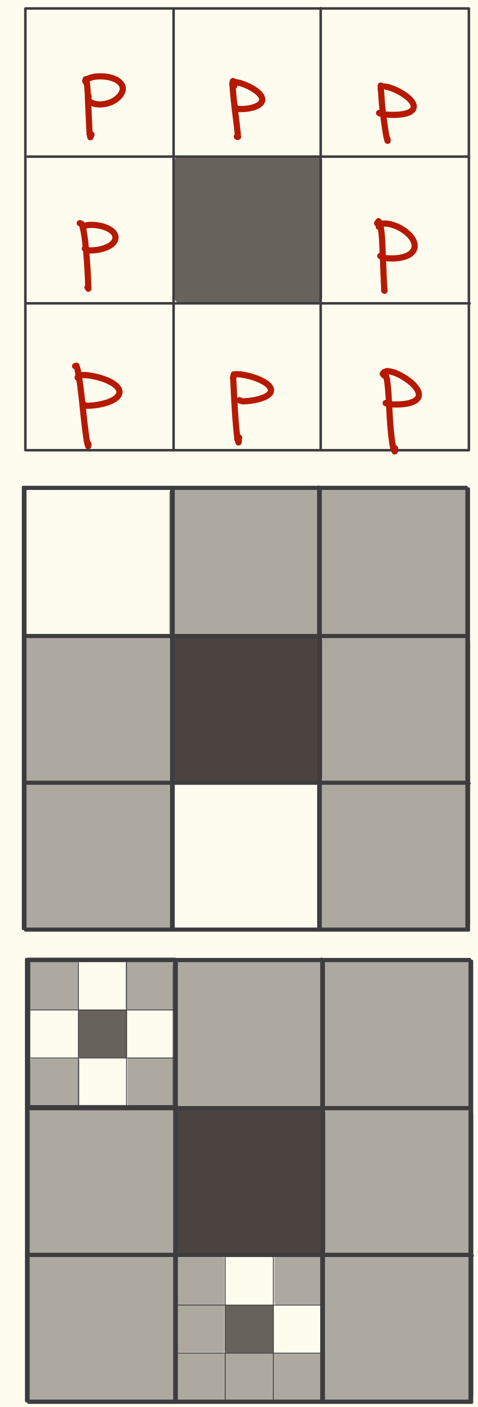
Exceptional Directions: Randomness

$p \in (0,1]$



Exceptional Directions: Randomness

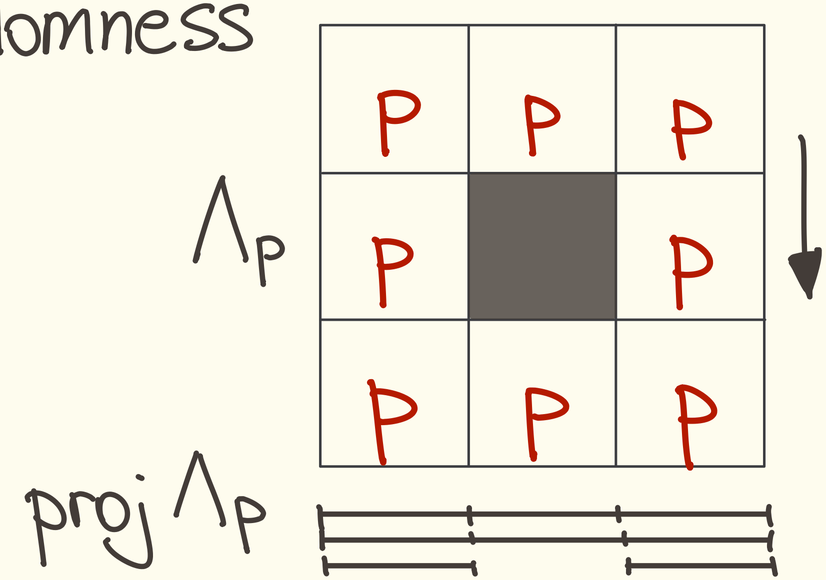
$p \in (0,1]$



$\dots \Lambda_p$

EXCEPTIONAL DIRECTIONS: Randomness

$p \in (0,1]$



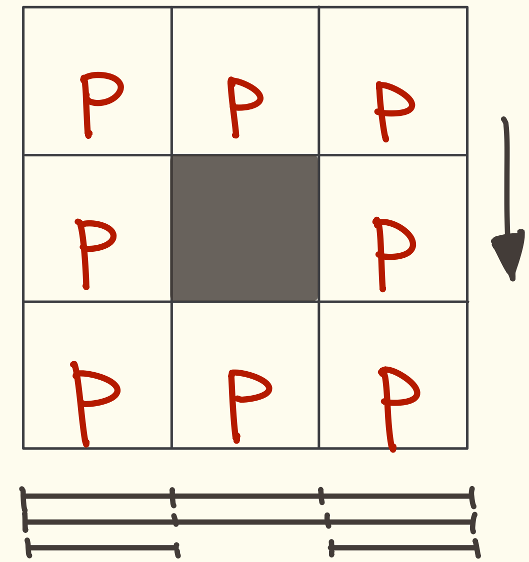
(1) orthogonal projections to the x-axis:

- Dekking–Grimmett '88
- Falconer '89
- Falconer–Grimmet '92

EXCEPTIONAL DIRECTIONS: Randomness

$p \in (0,1]$

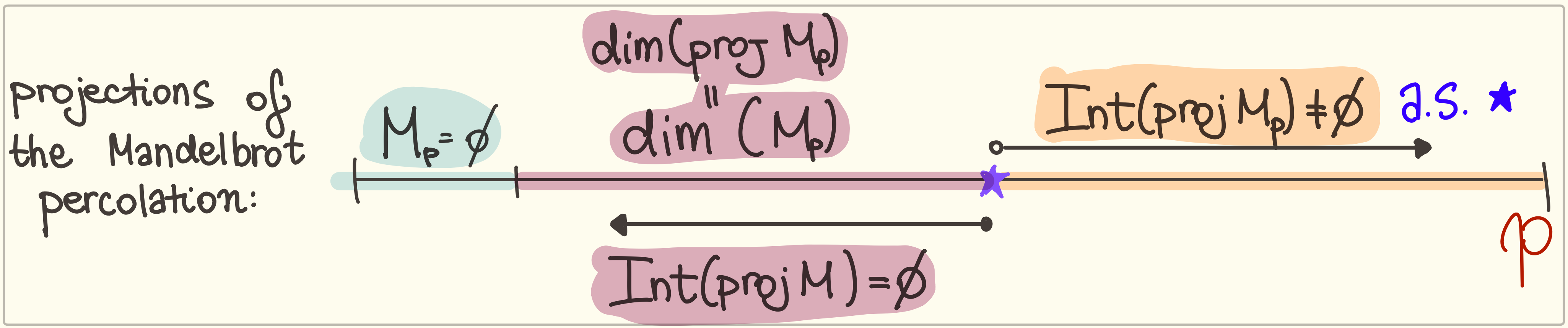
Λ_p

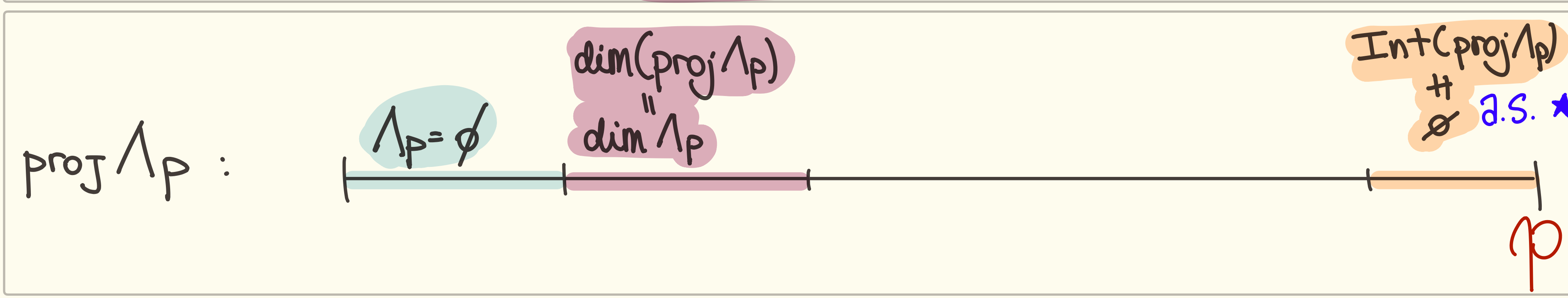


proj Λ_p

(1) orthogonal projections to the x-axis:

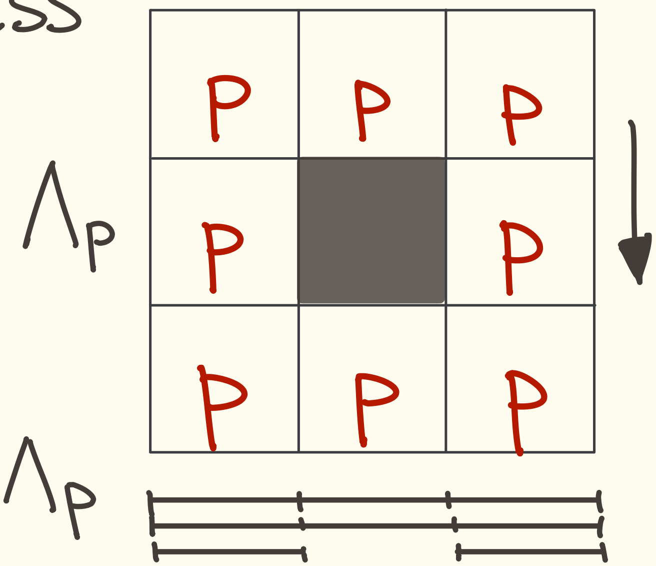
- Dekking-Grimmett '88
- Falconer '89
- Falconer-Grimmet '92





EXCEPTIONAL DIRECTIONS: Randomness

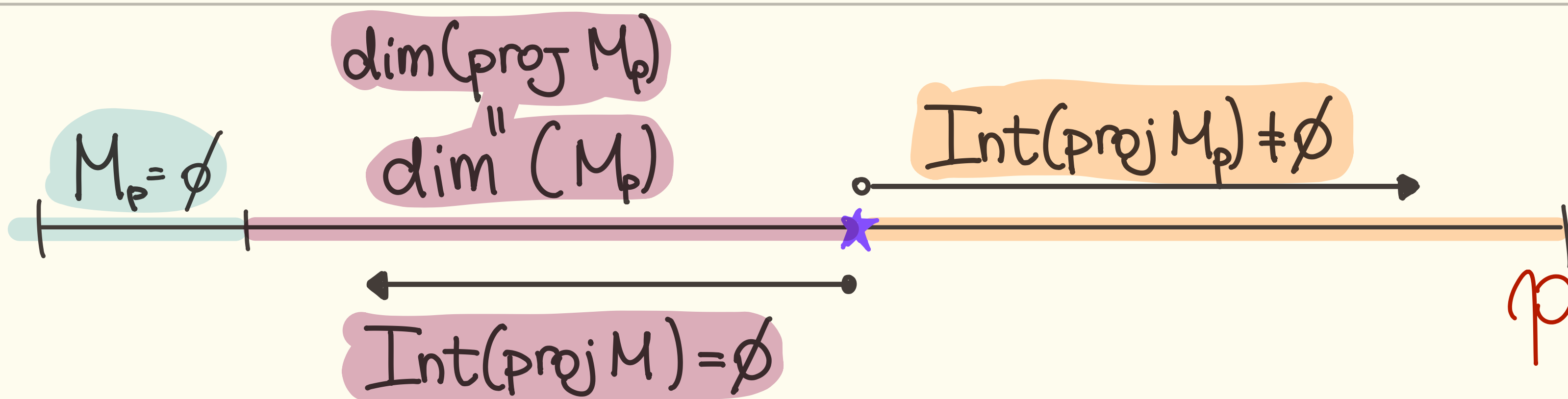
$p \in (0, 1]$



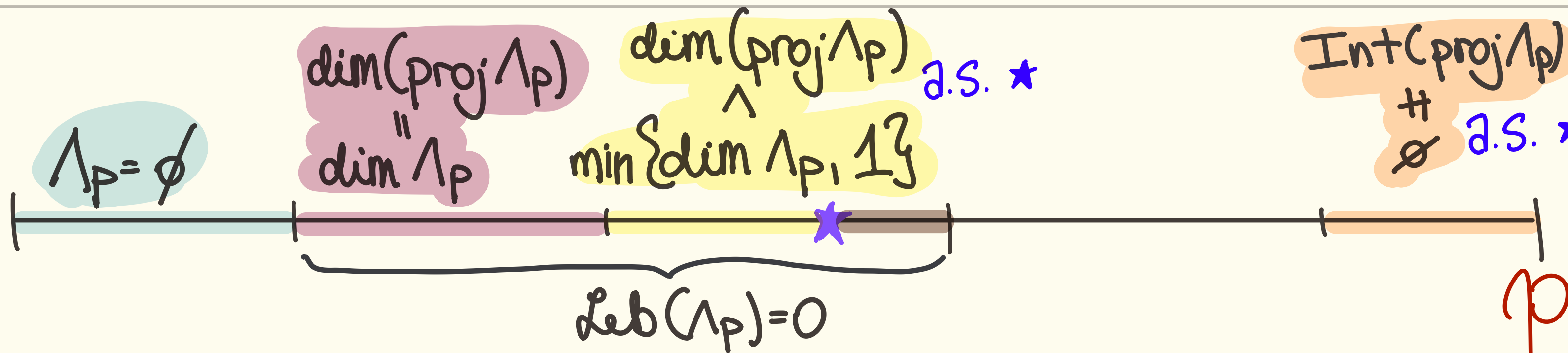
(1) orthogonal projections to the x-axis:

- Dekking-Grimmett '88
- Falconer '89
- Falconer-Grimmet '92

projections of the Mandelbrot percolation:



proj Λ_p :

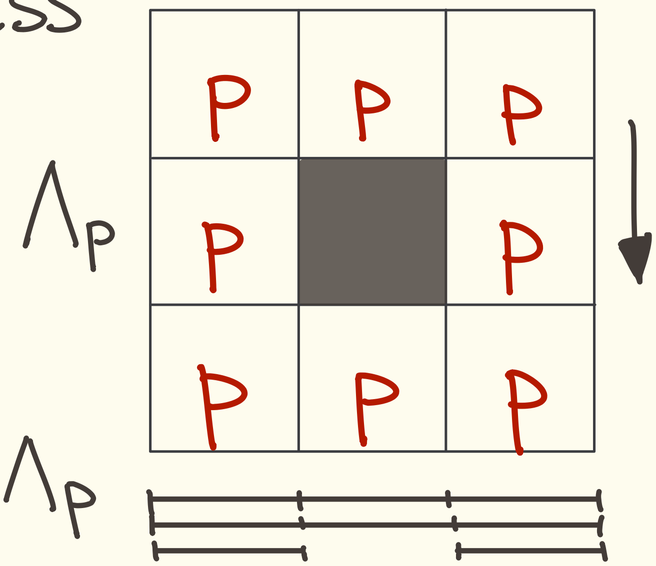


Exceptional Directions: Randomness

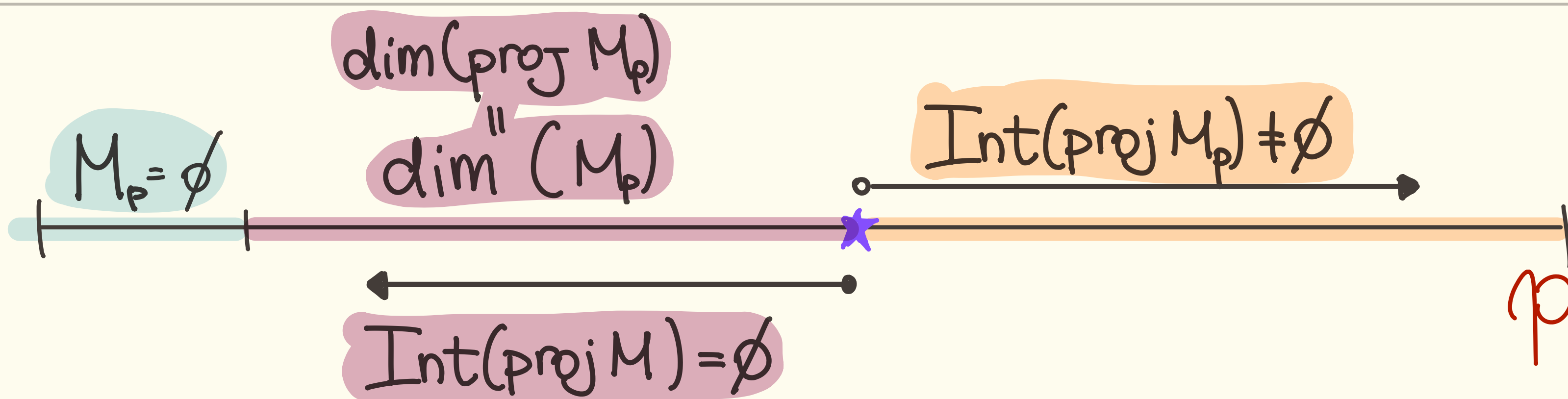
$p \in (0,1]$

(4) orthogonal projections to the x-axis:

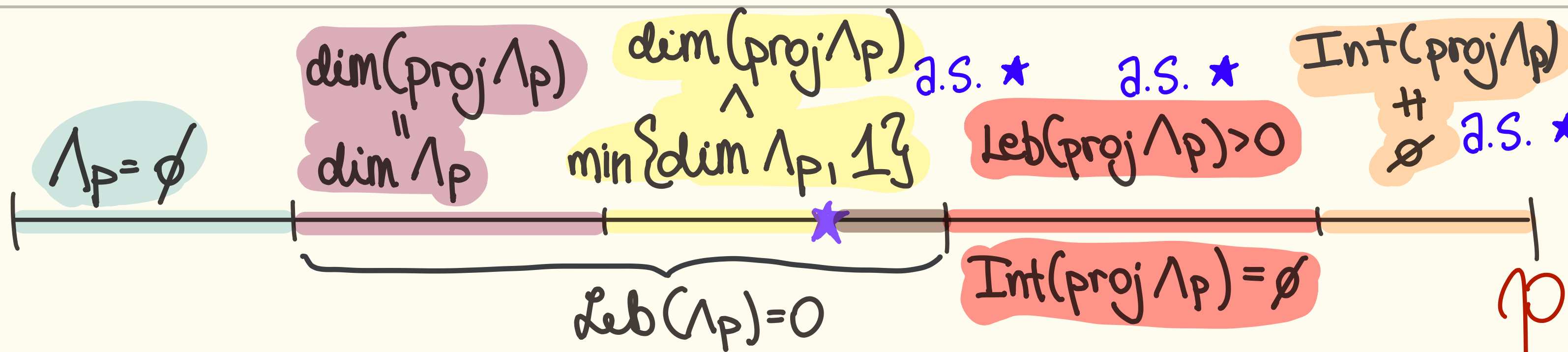
- Dekking–Grimmett '88
- Falconer '89
- Falconer–Grimmett '92



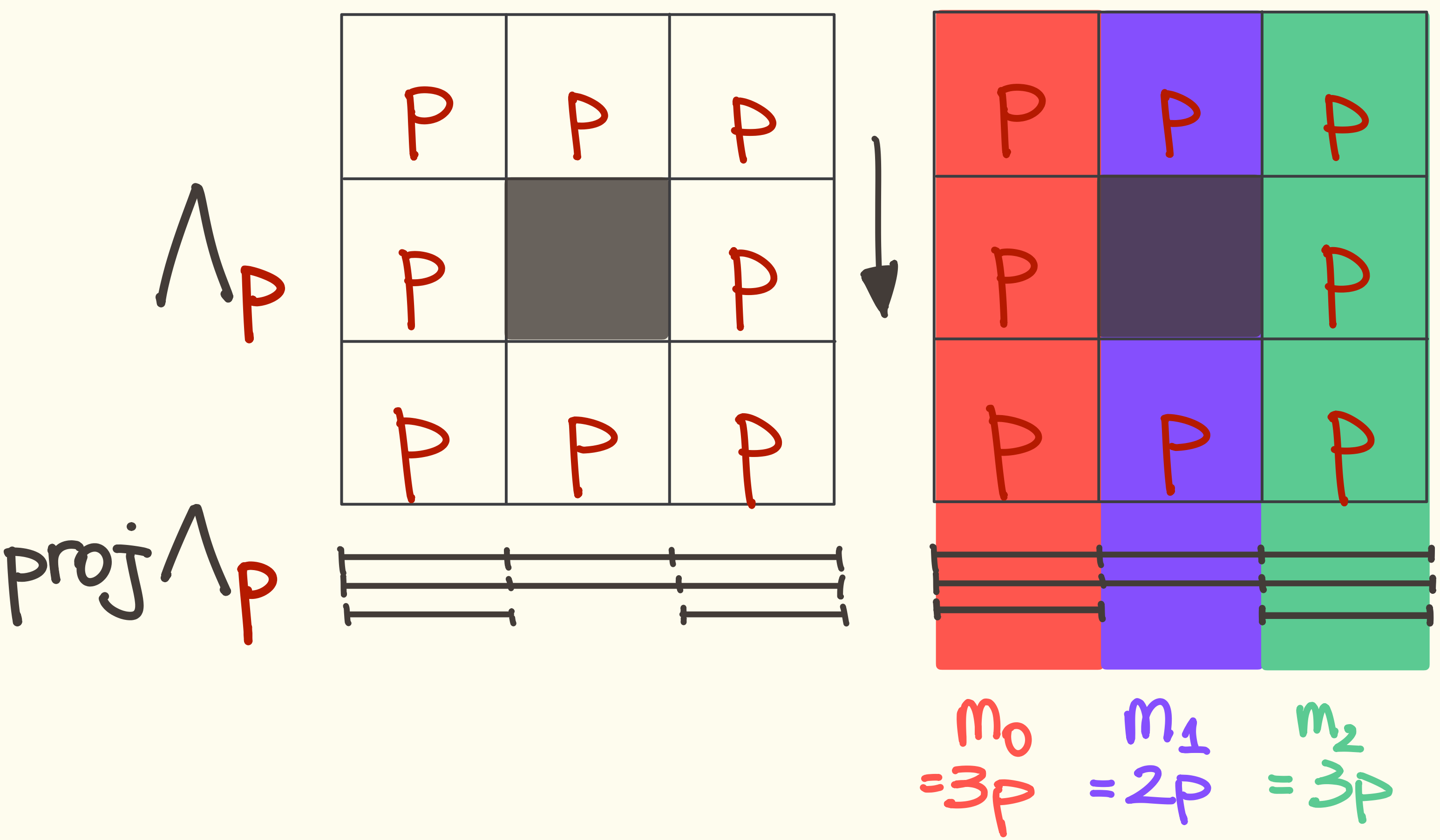
projections of the Mandelbrot percolation:



$\mathrm{proj}\,\Lambda_p$:

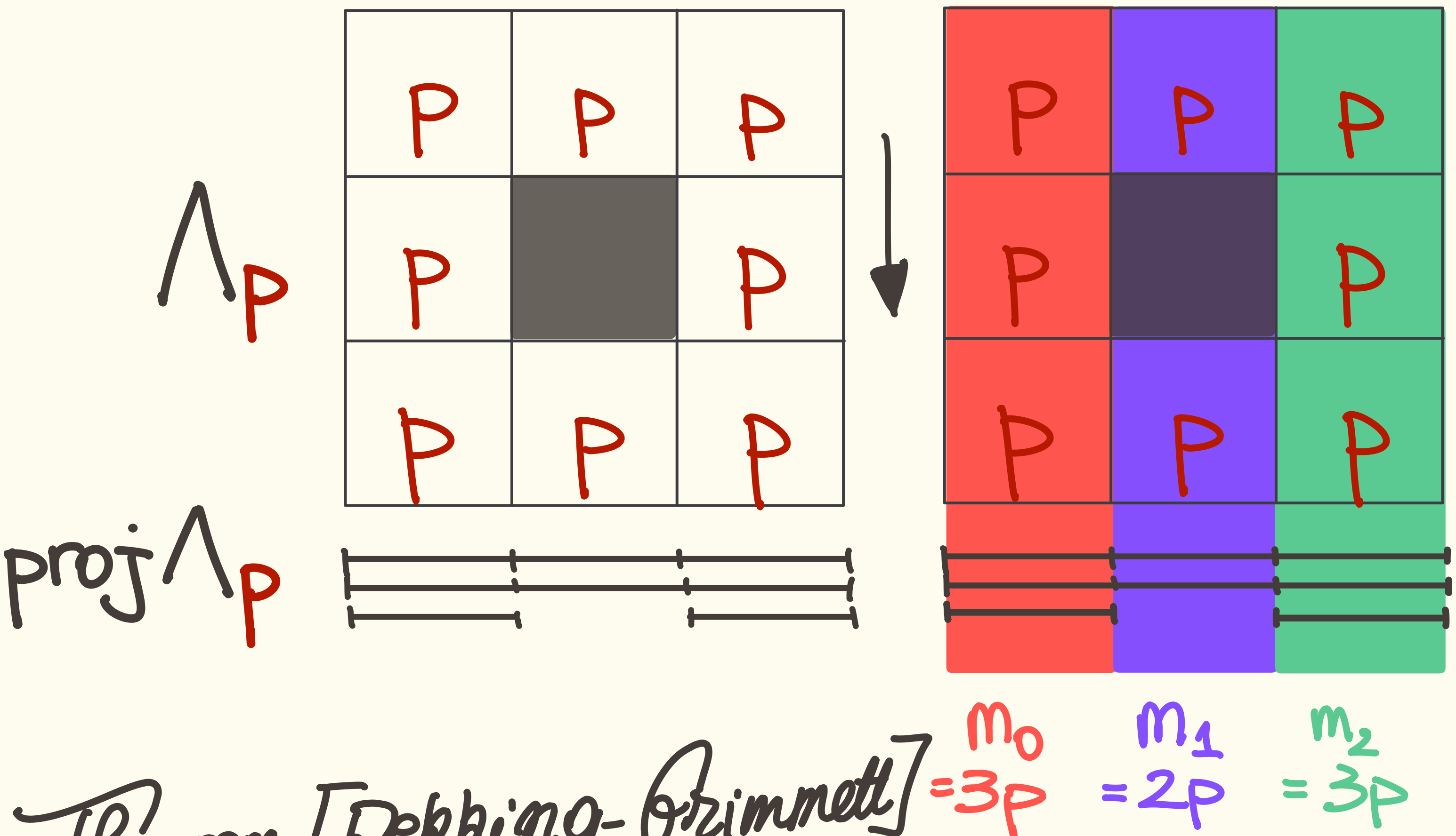


POSITIVITY OF LEBESGUE MEASURE



$$m = \prod_{i=0}^{L-1} m_i = m_0 m_1 m_2$$

Positivity of Lebesgue Measure

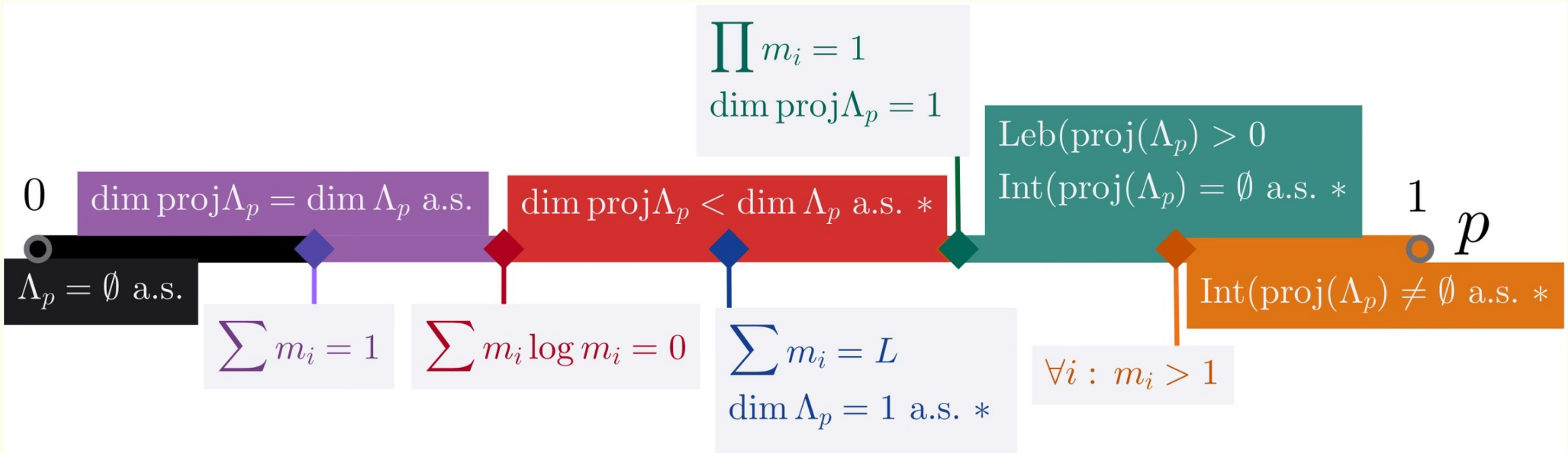


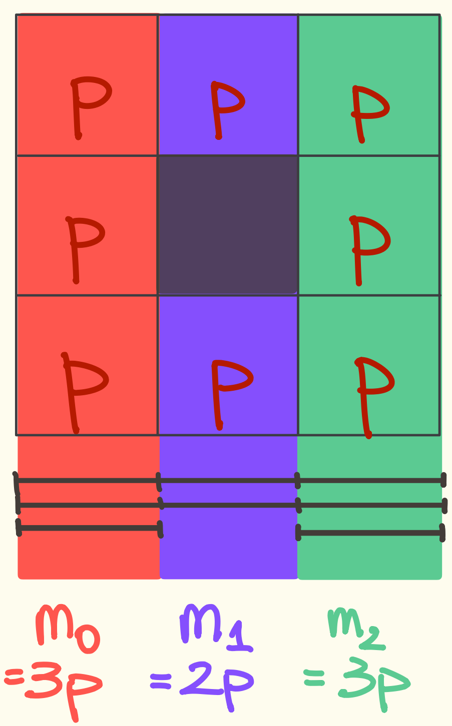
$$m = \prod_{i=0}^{L-1} m_i = m_0 m_1 m_2$$

Theorem [Dekking-Grimmett]

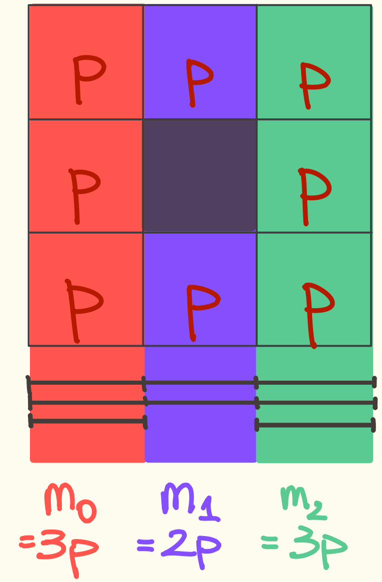
$\operatorname{Leb}(\operatorname{proj}\Lambda_P) > 0$ a.s. conditioned on non-extinction $\iff m = m_0 m_1 m_2 > 1$

Parameter Intervals





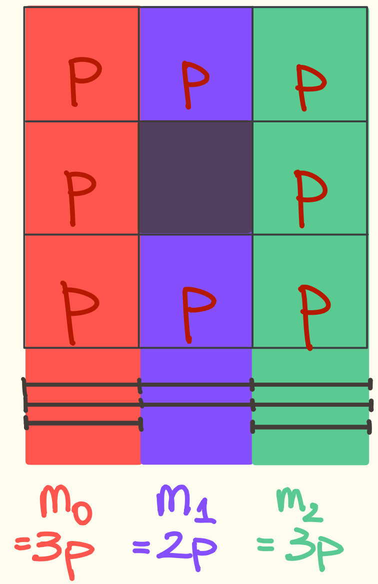
Leb(proj Λ_p) > 0 a.s. conditioned on non-extinction $\iff m > 0$



$$m = \prod_{i=0}^{L-1} m_i = m_0 m_1 m_2$$

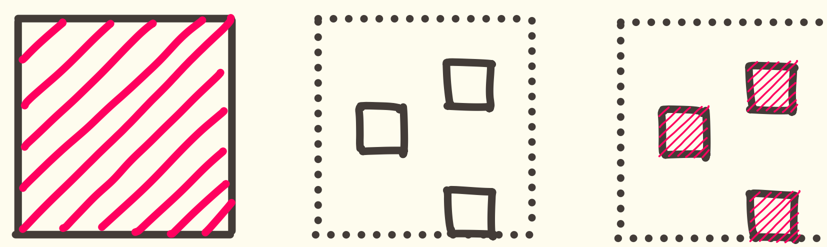
$\boxed{1.}$ Leb(proj Λ_p) > 0 a.s. conditioned on non-extinction $\iff \mathbb{P}(\text{Leb}(\text{proj}\,\Lambda_p) > 0) > 0$

$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$
a.s. conditioned on non-extinction $\iff m > 0$

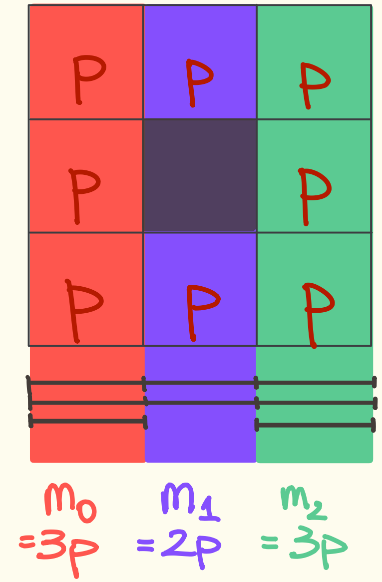


$$m = \prod_{i=0}^{L-1} m_i = m_0 m_1 m_2$$

$\boxed{1.}$ $\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$
a.s. conditioned on non-extinction $\iff \mathbb{P}\big(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0\big) > 0$

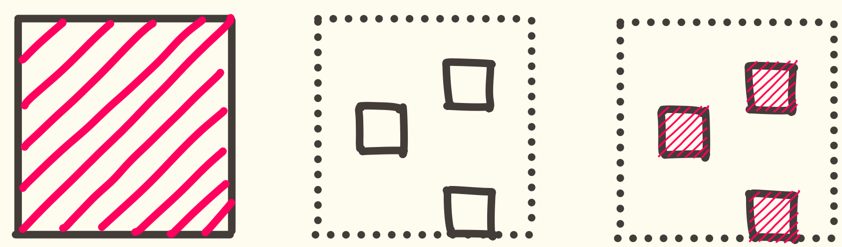


$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$ a.s. conditioned on non-extinction $\iff m > 0$



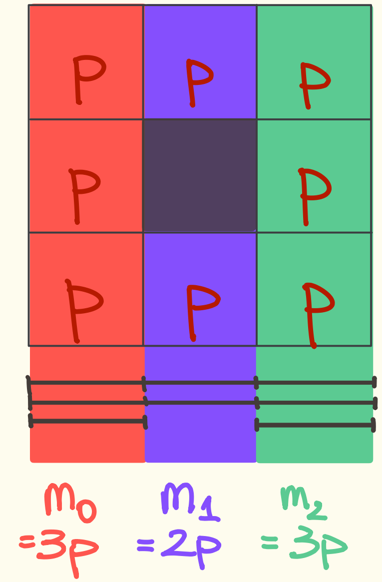
$$m = \prod_{i=0}^{L-1} m_i = m_0 m_1 m_2$$

$\boxed{1.}$ $\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$ a.s. conditioned on non-extinction $\iff \mathbb{P}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0) > 0$



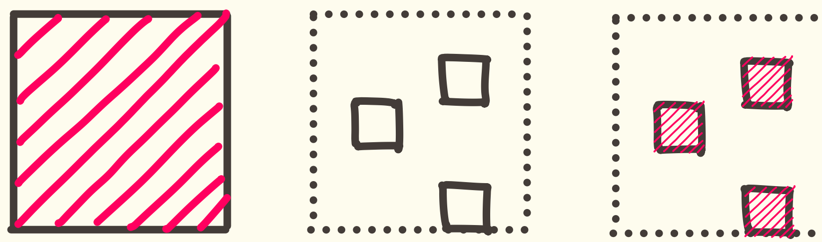
$\boxed{2.}$ $\mathbb{P}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0) > 0 \iff \mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) > 0$

$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$ a.s. conditioned on non-extinction $\iff m > 0$



$$m = \prod_{i=0}^{L-1} m_i = m_0 m_1 m_2$$

$\boxed{1.}$ $\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$ a.s. conditioned on non-extinction $\iff \mathbb{P}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0) > 0$

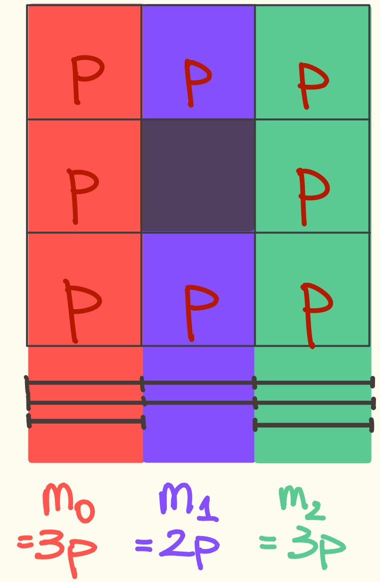


$\boxed{2.}$ $\mathbb{P}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0) > 0 \iff \mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) > 0$

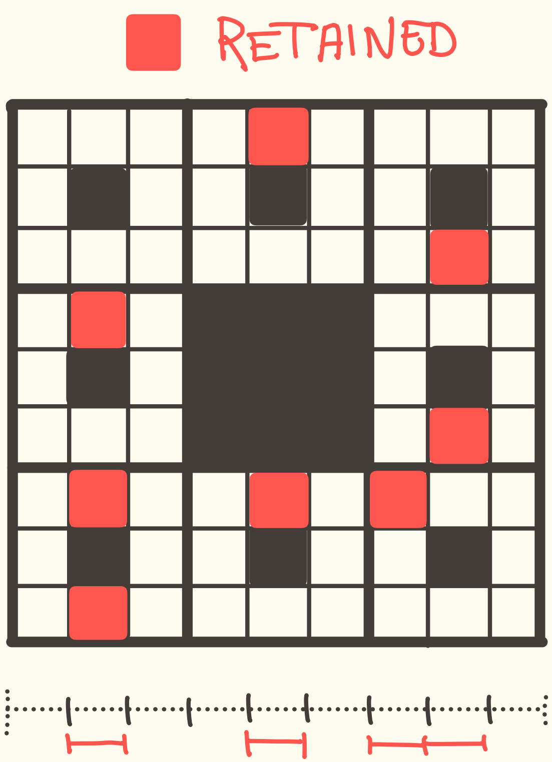
$\boxed{3.}$ $\mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) > 0 \iff \lim_{n\to\infty} \mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p^n)) > 0$

$$\lim_{n\to\infty} \mathbb{E}\left(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p^{n})\right) > 0 \iff m > 0$$

$$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p^{n}) = 3^{-n}\cdot \# Y_n$$



$$m = \prod_{i=0}^{L-1} m_i = m_0\, m_1\, m_2$$

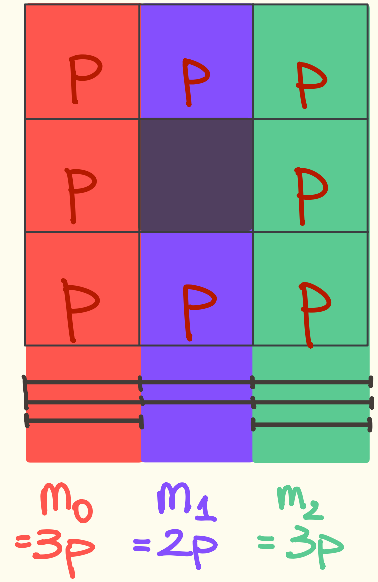


$$\{\underline{i} \in \{0,1,2\}^n : \text{the triadic interval corresponding to } \underline{i} \text{ is retained}\}$$

$$\overset{!!}{=}$$

$$Y_n$$

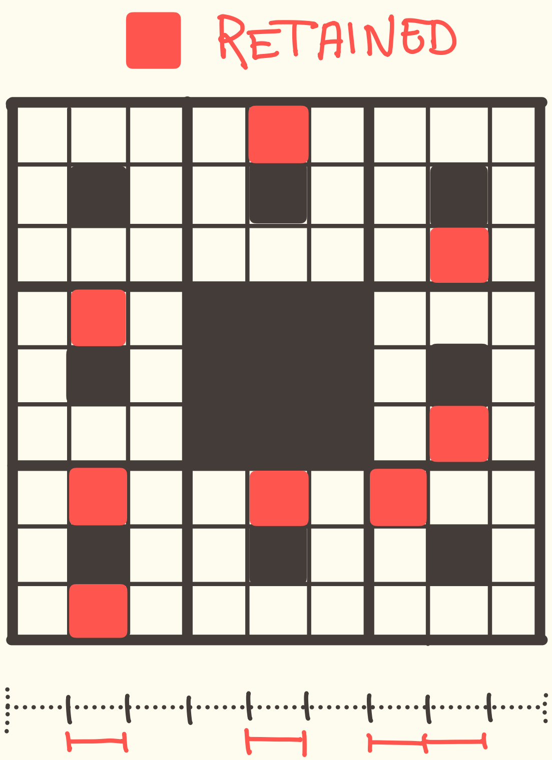
$$\lim_{n\to\infty} \mathbb{E}\left(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p^n)\right) > 0 \iff m > 0$$



$$m = \prod_{i=0}^{L-1} m_i = m_0 m_1 m_2$$

$$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p^n) = 3^{-n} \cdot \# Y_n$$

$$\mathbb{E}\left(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p^n)\right) = 3^{-n} \cdot \sum_{\underline{i}\in[3]^n} \mathbb{P}(\underline{i} \in Y_n)$$

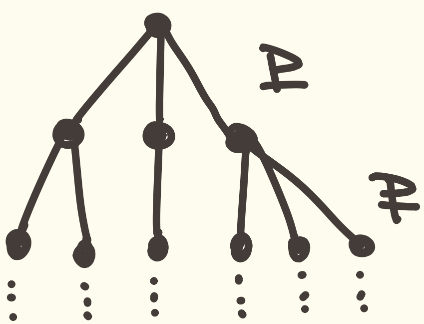


$$\{\underline{i} \in \{0,1,2\}^n : \text{the triadic interval corresponding to } \underline{i} \text{ is retained}\} = Y_n$$

KEY RELATIONSHIP 1. BRANCHING PROCESSES

G-W process

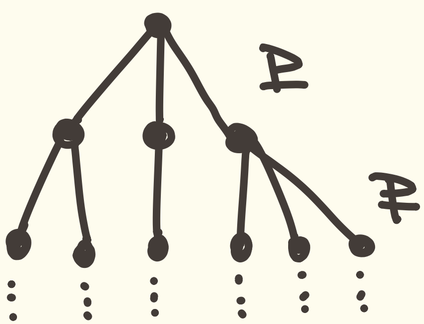
$\mathbb{P}$ offspring distn.



Z_n = # level n nodes

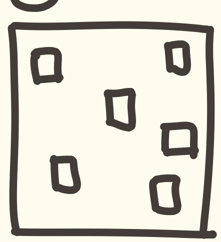
KEY RELATIONSHIP 1. BRANCHING PROCESSES

G-W process
p offspring distn.



Z_n = # level n nodes

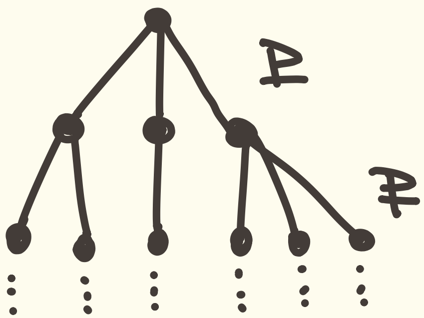
e.g.



retained level-n squares

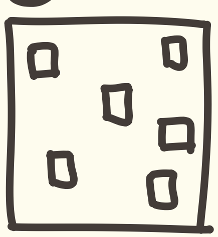
KEY RELATIONSHIP 1. BRANCHING PROCESSES

G-W process
$\mathbb{P}$ offspring distn.



Z_n = # level n nodes

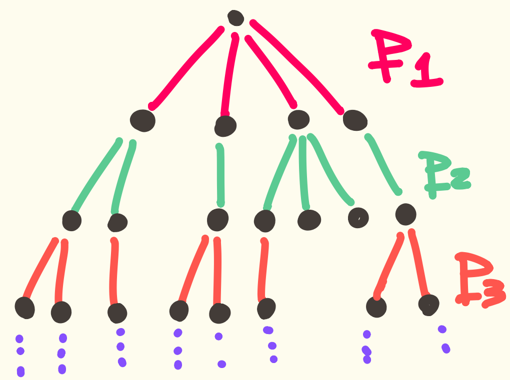
e.g.



retained level-n squares

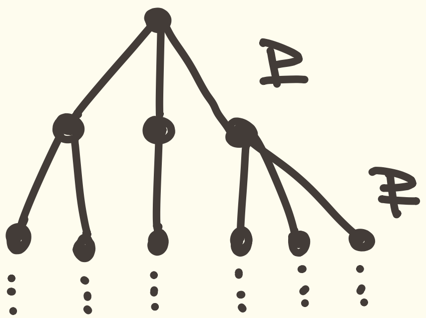
branching processes in varying environments

$(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4, \ldots)$



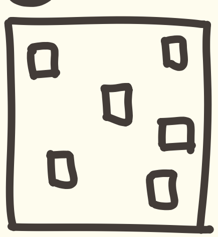
KEY RELATIONSHIP 1. BRANCHING PROCESSES

G-W process
P offspring dist'n.



Z_n = # level n nodes

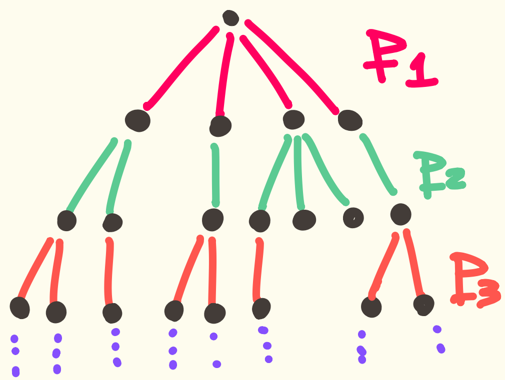
e.g.



retained level-n squares

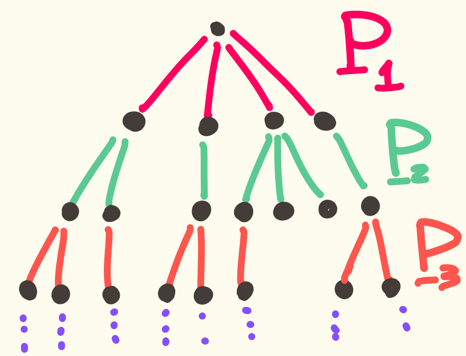
branching processes in varying environments

$(P_1, P_2, P_3, P_4, \dots)$



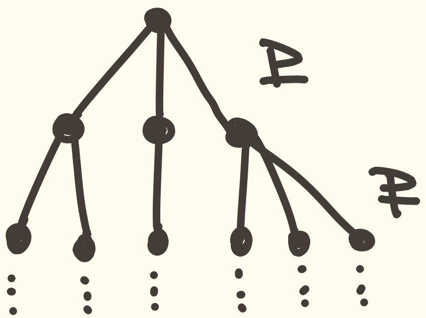
branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \dots)$ is a random sequence of distributions.



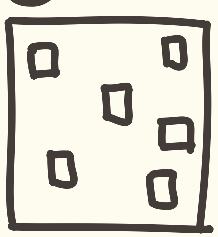
KEY RELATIONSHIP 1. BRANCHING PROCESSES

G-W process
P offspring distn.



Z_n = # level n nodes

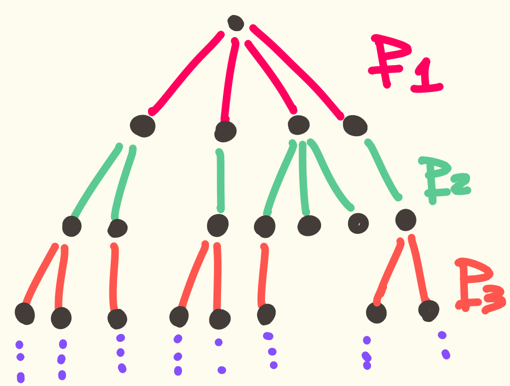
e.g.



retained level-n squares

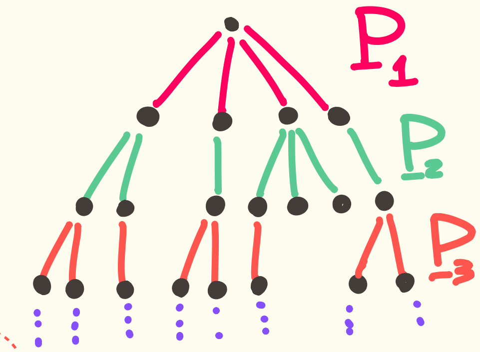
branching processes in varying environments

$(P_1, P_2, P_3, P_4, \dots)$



branching processes in RANDOM environments

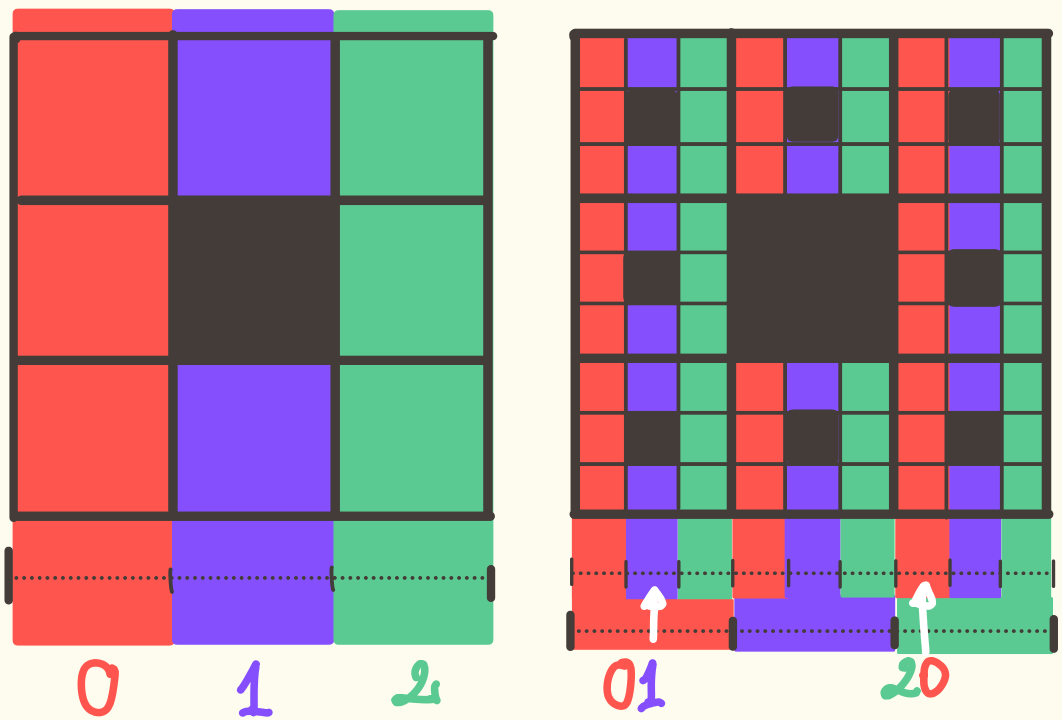
$(P_1, P_2, P_3, P_4, \dots)$ is a random sequence of distributions.



e.g. $(\underset{1/3}{q_1}, \underset{1/3}{q_2}, \underset{1/3}{q_3})$ distr.

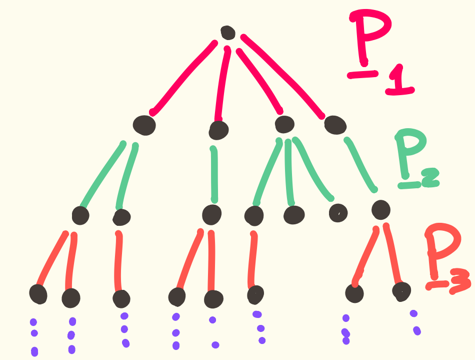
$$(\underset{1.}{\cdot}, \underset{2.}{\cdot}, \underset{3.}{\cdot}, \underset{4.}{\cdot}, \underset{\dots}{\dots})$$

KEY RELATIONSHIP 1. BPREs



branching processes in RANDOM environments

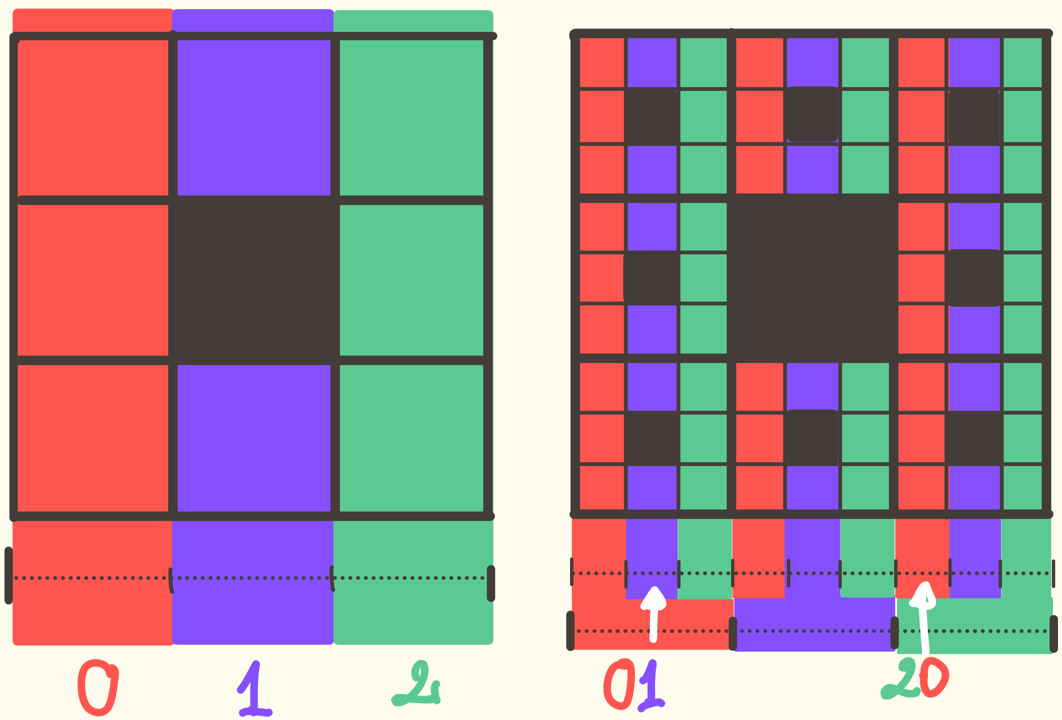
($P_1, P_2, P_3, P_4, \dots$) is a random seq. of distr.



$$\big(\underbrace{Bin(3,p)}_{\varphi_0},\ \underbrace{Bin(2,p)}_{\varphi_1},\ \underbrace{Bin(3p)}_{\varphi_2}\big)$$

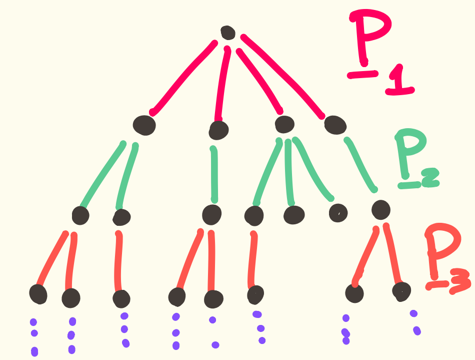
$\mathbb{P}(\varphi_0) = 1/3 \quad \mathbb{P}(\varphi_1) = 1/3 \quad \mathbb{P}(\varphi_2) = 1/3$

KEY RELATIONSHIP 1. BPREs



branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \dots)$ is a random seq. of distr.

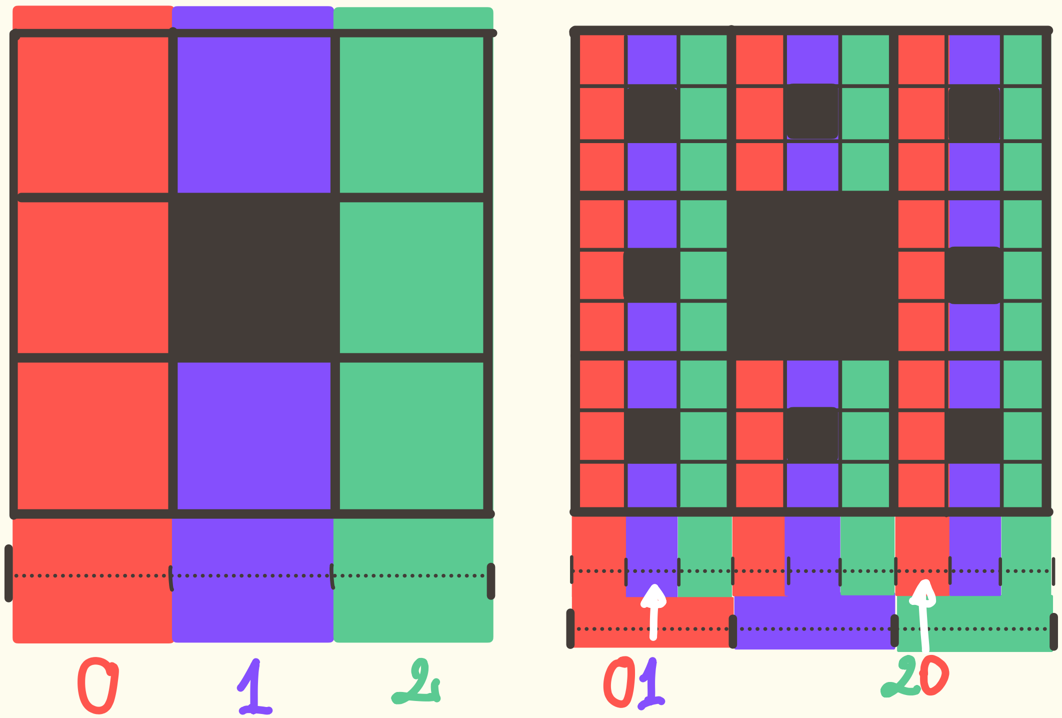


$(i_1, \dots, i_n) = \underline{i} \in \{0,1,2\}^n$

$$\left[\begin{array}{ccc} (\mathrm{Bin}(3,p), & \mathrm{Bin}(2,p), & \mathrm{Bin}(3p)) \\ q_0 & q_1 & q_2 \\ P(q_0) = 1/3 & P(q_1) = 1/3 & P(q_2) = 1/3 \end{array}\right]$$

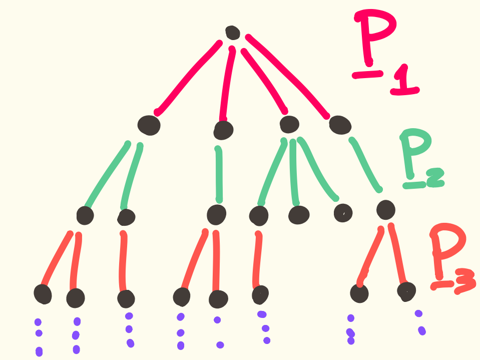
$\tilde{Z}_n(\underline{i}) = \#\{$num of level-n children given $q_{i_1}, \dots, q_{i_n}\}$

KEY RELATIONSHIP 1. BPREs



branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \dots)$ is a random seq. of distr.



$(i_1, \dots, i_n) = \underline{i} \in \{0,1,2\}^n$

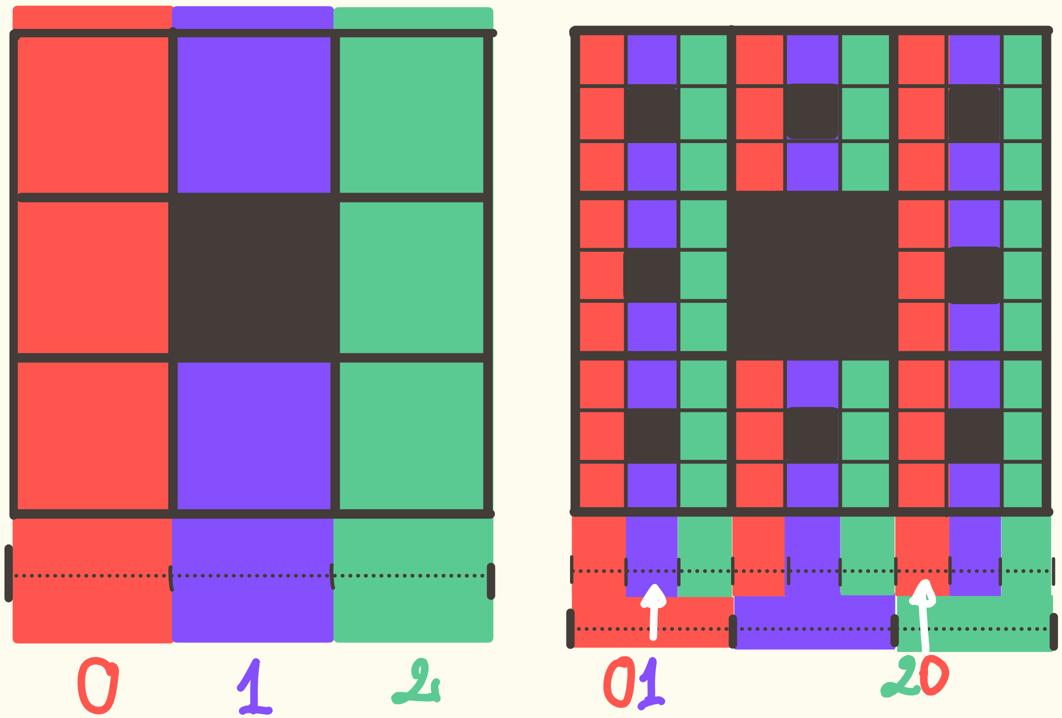
$$\big(\underset{q_0}{\mathrm{Bin}(3,p)}, \underset{q_1}{\mathrm{Bin}(2,p)}, \underset{q_2}{\mathrm{Bin}(3p)}\big)$$

$P(q_0) = 1/3 \quad P(q_1) = 1/3 \quad P(q_2) = 1/3$

$\tilde{Z}_n(\underline{i}) = \#\{$num of level-n children given $q_{i_1}, \dots, q_{i_n}\}$

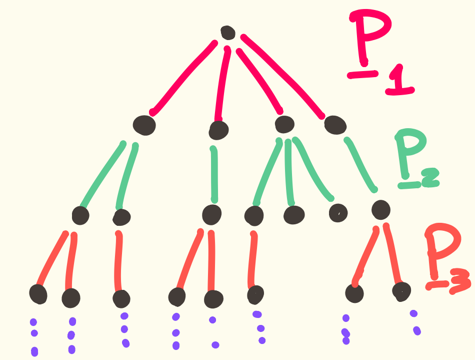
$\tilde{Z}_n = \#\{$num of level-n children$\}$

KEY RELATIONSHIP 1. BPREs



branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \dots)$ is a random seq. of distr.



$(i_1, \dots, i_n) = \underline{i} \in \{0,1,2\}^n$

$$\big(\mathrm{Bin}(3,p), \mathrm{Bin}(2,p), \mathrm{Bin}(3p)\big)$$

$$f_0 \qquad f_1 \qquad f_2$$

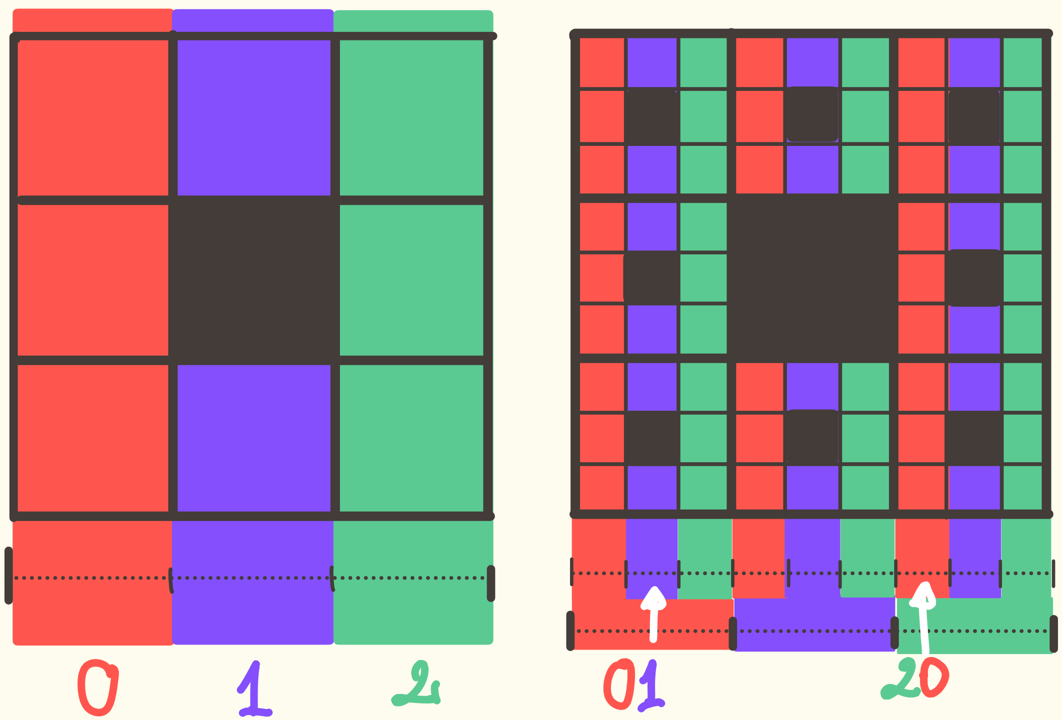
$$\mathbb{P}(f_0) = 1/3 \qquad \mathbb{P}(f_1) = 1/3 \qquad \mathbb{P}(f_2) = 1/3$$

$\tilde{Z}_n(\underline{i}) = \#\{\text{num of level-n children given } f_{i_1}, \dots, f_{i_n}\}$

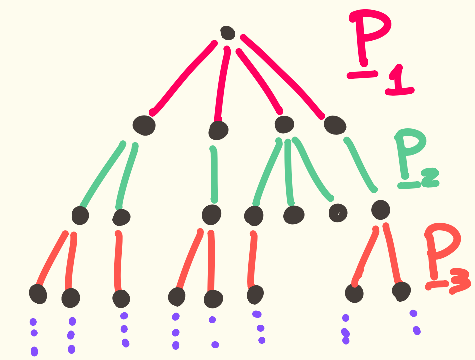
$\tilde{Z}_n = \#\{\text{num of level-n children}\}$

$$3^{-n} \sum_{\underline{i} \in [3]^n} \mathbb{P}(\underline{i} \in Y_n) = \frac{1}{3^n} \sum_{\underline{i} \in [3]^n} \mathbb{P}(\tilde{Z}_n(\underline{i}) > 0)$$

KEY RELATIONSHIP 1. BPREs



branching processes in RANDOM environments
$(P_1, P_2, P_3, P_4, \dots)$ is a random seq. of distr.



$(i_1, \dots, i_n) = \underline{i} \in \{0,1,2\}^n$

$$\big(\mathrm{Bin}(3,p), \mathrm{Bin}(2,p), \mathrm{Bin}(3p)\big)$$

φ_0 $\quad$ φ_1 $\quad$ φ_2

$\mathbb{P}(\varphi_0) = 1/3 \quad \mathbb{P}(\varphi_1) = 1/3 \quad \mathbb{P}(\varphi_2) = 1/3$

$\tilde{Z}_n(\underline{i}) = \#\{$num of level-n children given $\varphi_{i_1}, \dots, \varphi_{i_n}\}$

$\tilde{Z}_n = \#\{$num of level-n children$\}$

$$3^{-n} \sum_{\underline{i} \in [3]^n} \mathbb{P}(\underline{i} \in Y_n) = \frac{1}{3^n} \sum_{\underline{i} \in [3]^n} \mathbb{P}(\tilde{Z}_n(\underline{i}) > 0) = \mathbb{P}(\tilde{Z}_n > 0)$$

g

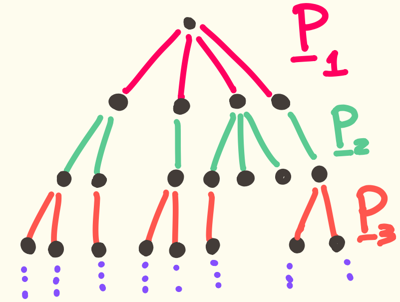
SURVIVAL OF BPREs

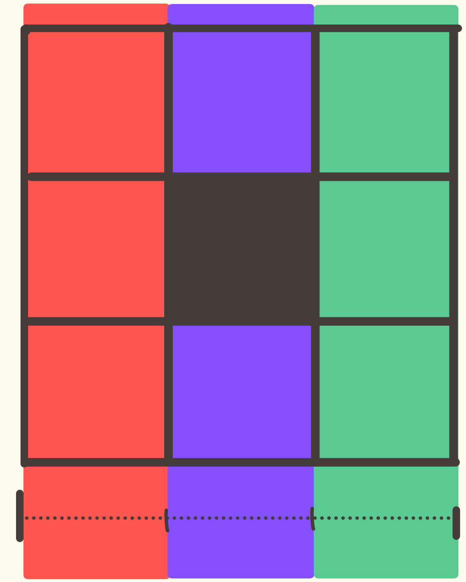
$\tilde{Z}_n$ is a branching process in a random environment.

QUESTION: $\lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0)$?

branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \dots)$ is a random seq. of distr.





$\tilde{Z}_n$ b.p.r.e.

$$\left[\begin{array}{ccc} (\mathrm{Bin}(3,p), & \mathrm{Bin}(2,p), & \mathrm{Bin}(3p)) \\ P_0 & P_1 & P_2 \\ \mathbb{P}(P_0) = 1/3 & \mathbb{P}(P_1) = 1/3 & \mathbb{P}(P_2) = 1/3 \\ \mathbb{E}(P_0) = 3p & \mathbb{E}(P_1) = 2p & \mathbb{E}(P_2) = 3p \end{array}\right.$$

SURVIVAL OF BPREs

$\tilde{Z}_n$ is a branching process in a random environment.

QUESTION: $\lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0)$?

THEOREM [ATHREYA-KARLIN] '71:

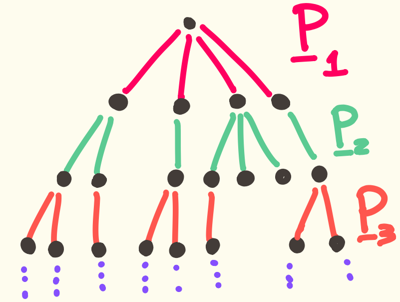
$$\lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0) > 0$$

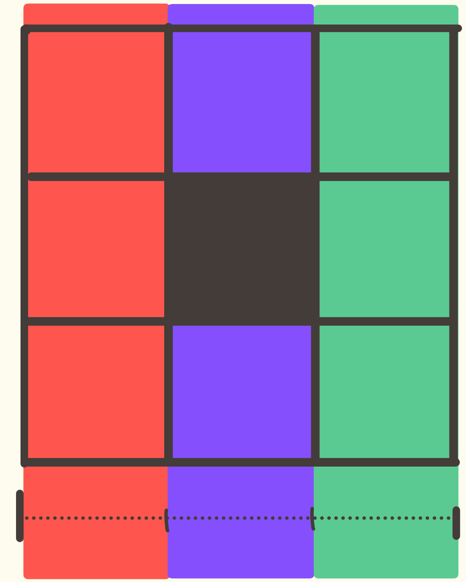
$$\Updownarrow$$

$$m = \mathbb{E}(P_0)\cdot\mathbb{E}(P_1)\cdot\mathbb{E}(P_2) > 1$$

branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \ldots)$ is a random seq. of distr.





$\tilde{Z}_n$ b.p.r.e.

$(\mathrm{Bin}(3,p), \mathrm{Bin}(2,p), \mathrm{Bin}(3p))$

P_0 P_1 P_2

$\mathbb{P}(P_0) = 1/3$ $\mathbb{P}(P_1) = 1/3$ $\mathbb{P}(P_2) = 1/3$

$\mathbb{E}(P_0) = 3p$ $\mathbb{E}(P_1) = 2p$ $\mathbb{E}(P_2) = 3p$

SURVIVAL OF BPREs

$\tilde{Z}_n$ is a branching process in a random environment.

QUESTION: $\lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0)$?

THEOREM [ATHREYA-KARLIN] '71:

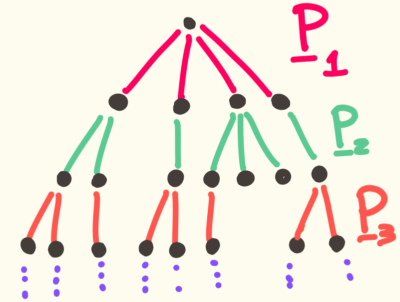
$$\lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0) > 0$$

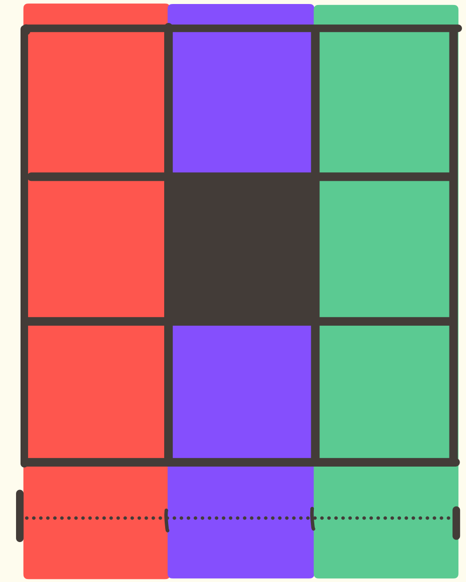
$$\Updownarrow$$

$$m = \mathbb{E}(P_0)\cdot\mathbb{E}(P_1)\cdot\mathbb{E}(P_2) > 1$$

branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \dots)$ is a random seq. of distr.





$\tilde{Z}_n$ b.p.r.e.

$(\mathrm{Bin}(3,p),\ \mathrm{Bin}(2,p),\ \mathrm{Bin}(3,p))$

$P_0 \quad P_1 \quad P_2$

$\mathbb{P}(P_0) = 1/3 \quad \mathbb{P}(P_1) = 1/3 \quad \mathbb{P}(P_2) = 1/3$

$\mathbb{E}(P_0) = 3p \quad \mathbb{E}(P_1) = 2p \quad \mathbb{E}(P_2) = 3p$

Summary:

$$\mathbb{P}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0) > 0 \Leftrightarrow \mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) > 0$$

$$\mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) = \lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0)$$

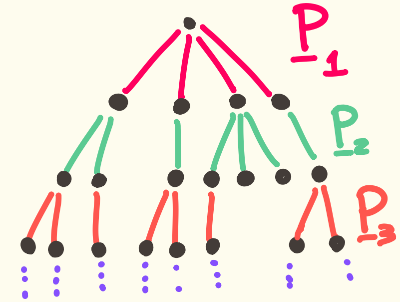
SURVIVAL OF BPREs

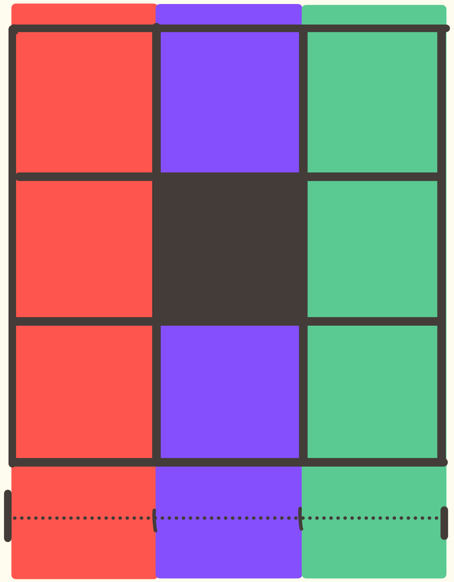
$\tilde{Z}_n$ is a branching process in a random environment.

QUESTION: $\lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0)$?

branching processes in RANDOM environments

$(P_1, P_2, P_3, P_4, \ldots)$ is a random seq. of distr.





THEOREM [ATHREYA-KARLIN]:

$$\lim_{n\to\infty} \mathbb{P}(\tilde{Z}_n > 0) > 0$$

$$\Updownarrow$$

$$m = \mathbb{E}(P_0) \cdot \mathbb{E}(P_1) \cdot \mathbb{E}(P_2) > 1$$

$\tilde{Z}_n$ b.p.r.e.

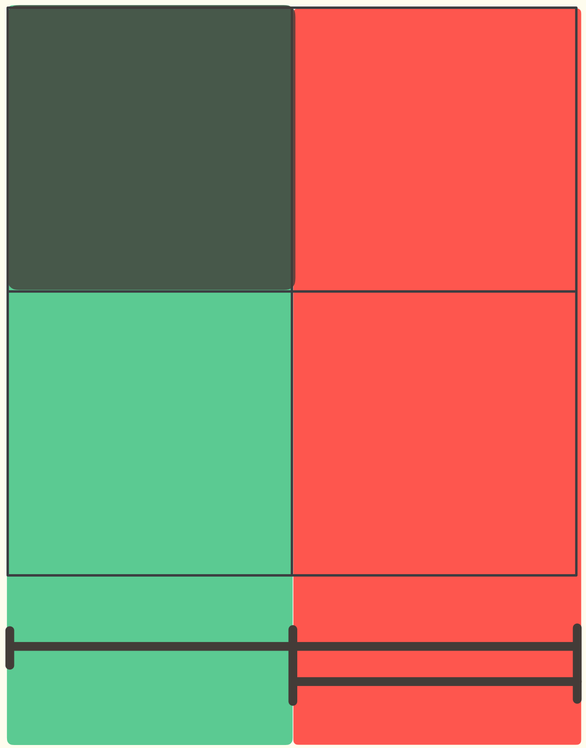
$$\left[\begin{array}{ccc} (\mathrm{Bin}(3,p), & \mathrm{Bin}(2,p), & \mathrm{Bin}(3p)) \\ P_0 & P_1 & P_2 \\ \mathbb{P}(P_0) = 1/3 & \mathbb{P}(P_1) = 1/3 & \mathbb{P}(P_2) = 1/3 \\ \mathbb{E}(P_0) = 3p & \mathbb{E}(P_1) = 2p & \mathbb{E}(P_2) = 3p \end{array}\right.$$

Summary:

$$\mathbb{P}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0) > 0 \iff \mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) > 0$$

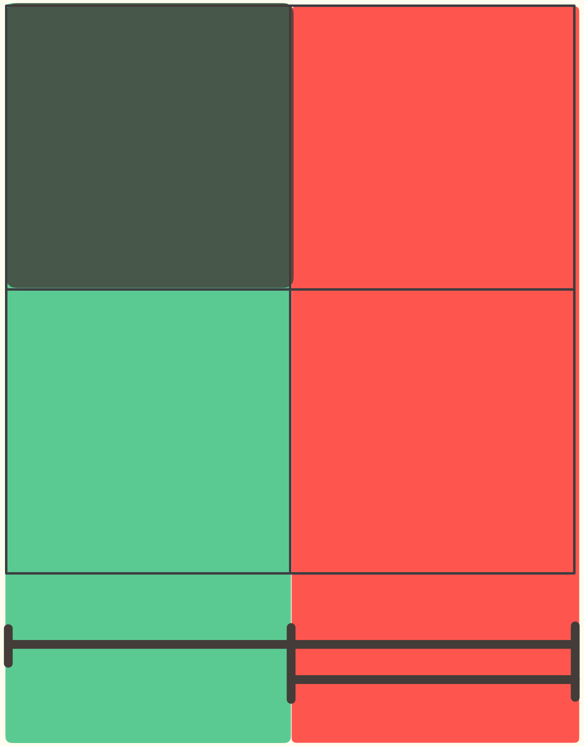
$$\mathbb{E}(\mathrm{Leb}(\mathrm{proj}\,\Lambda_p)) = \mathbb{P}(\tilde{Z}_n > 0)$$

RATIONAL PROJ.



$m_0 = p$, $\quad m_1 = 2p$

RATIONAL PROJ.



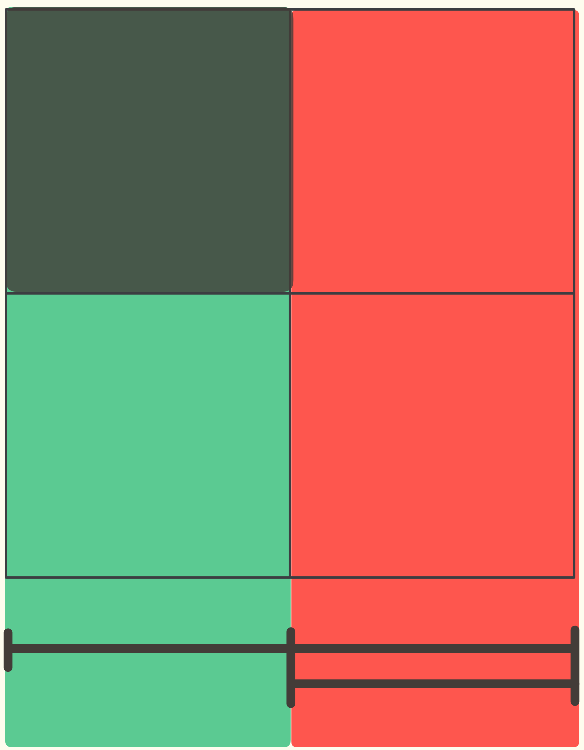
$m_0 = p$, $\quad m_1 = 2p$

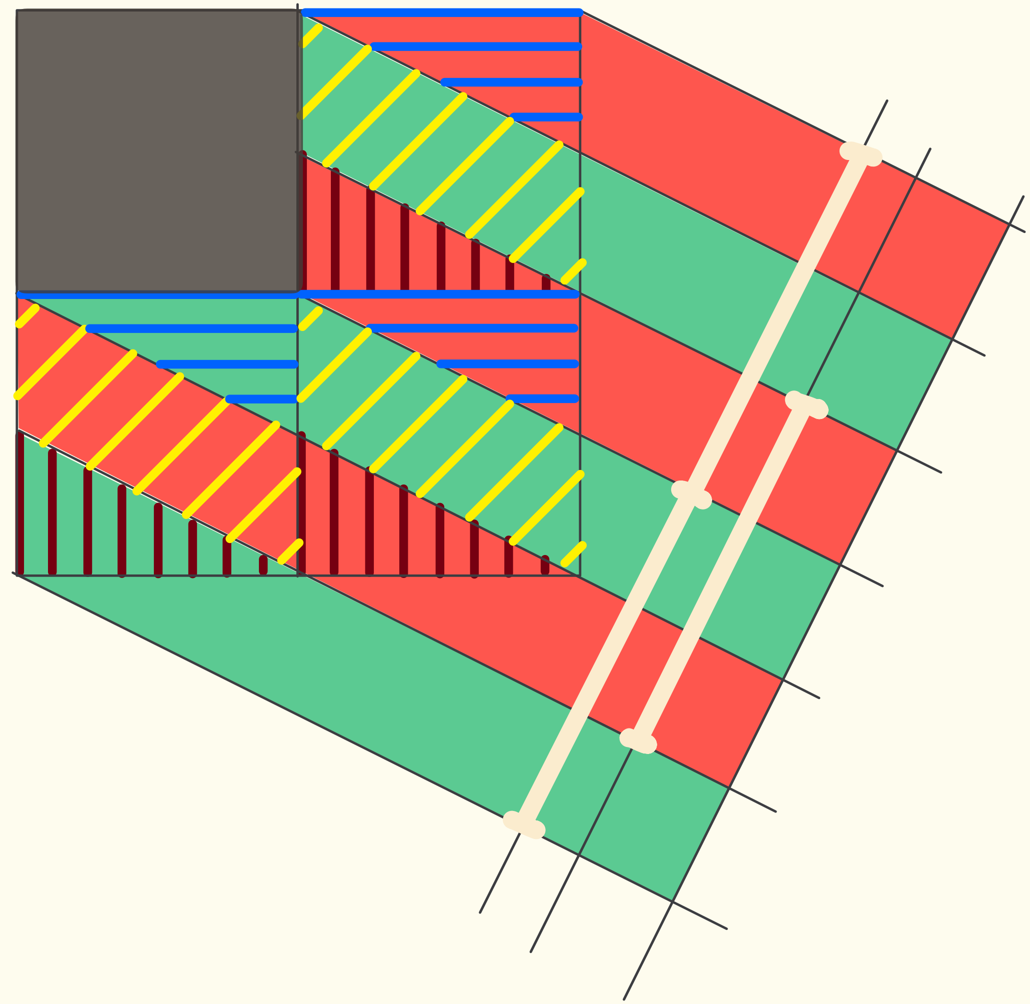
$\log m_0 \cdot m_1 > 0 \iff$

$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$

branching process in a random environment

RATIONAL PROJ.





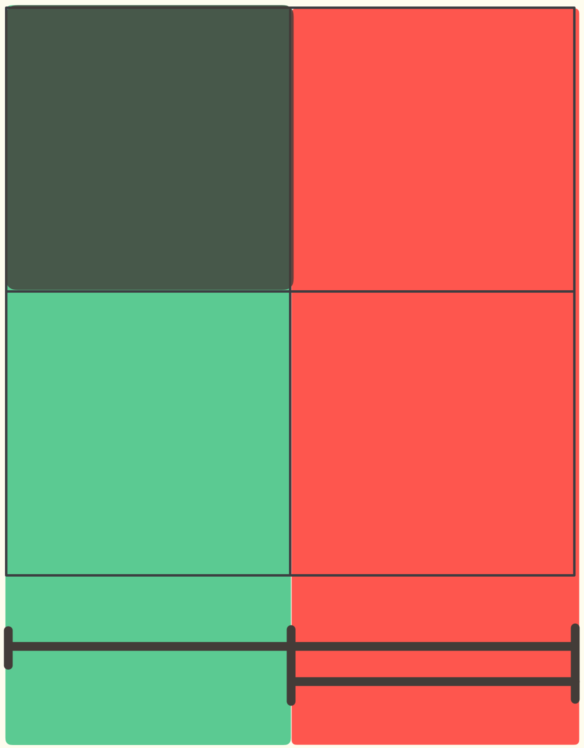
$m_0 = p$, $m_1 = 2p$

$\log m_0 \cdot m_1 > 0 \iff$

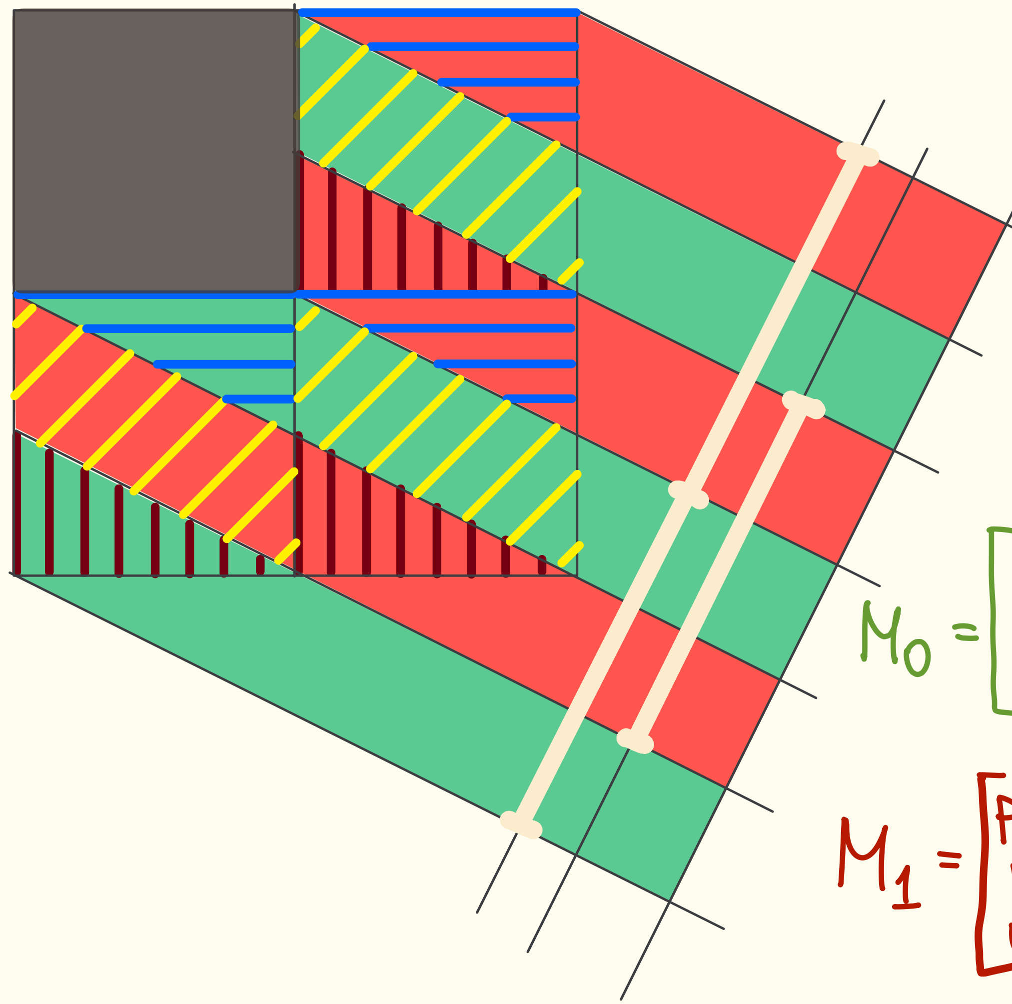
$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$

branching process in a random environment

RATIONAL PROJ.



$m_0 = p$, $\quad m_1 = 2p$



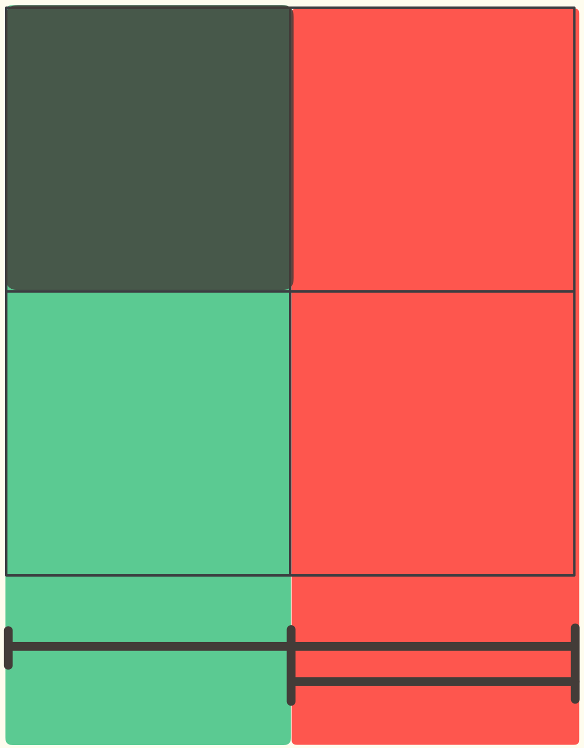
$M_0 = \begin{bmatrix} p & 0 & 0 \\ 0 & p & p \\ 0 & p & 0 \end{bmatrix}$, $\quad M_1 = \begin{bmatrix} p & p & 0 \\ p & 0 & p \\ 0 & 0 & p \end{bmatrix}$

$\log m_0 \cdot m_1 > 0 \iff$

$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$

branching process in a random environment

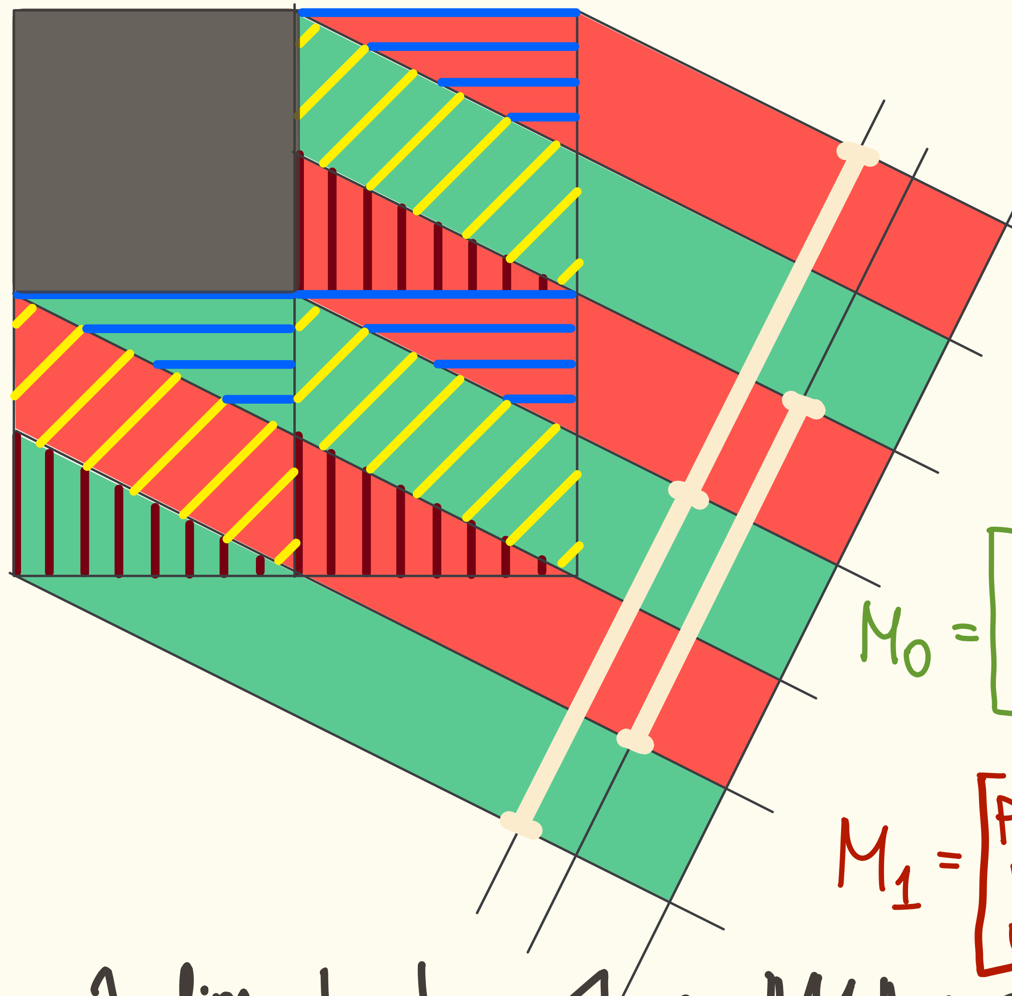
RATIONAL PROJ.



$m_0 = p$, $m_1 = 2p$

$$\log m_0 \cdot m_1 > 0 \iff \mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$$

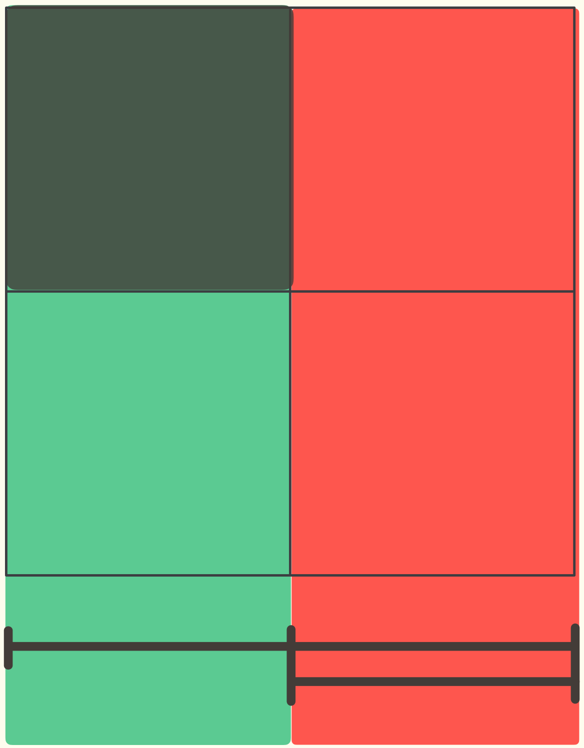
branching process in a random environment



$$M_0 = \begin{bmatrix} p & 0 & 0 \\ 0 & p & p \\ 0 & p & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} p & p & 0 \\ p & 0 & p \\ 0 & 0 & p \end{bmatrix}$$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{L^n} \cdot \sum_{\underline{i} \in [L]^n} \log \|M_{\underline{i}}\| > 0$$

RATIONAL PROJ.

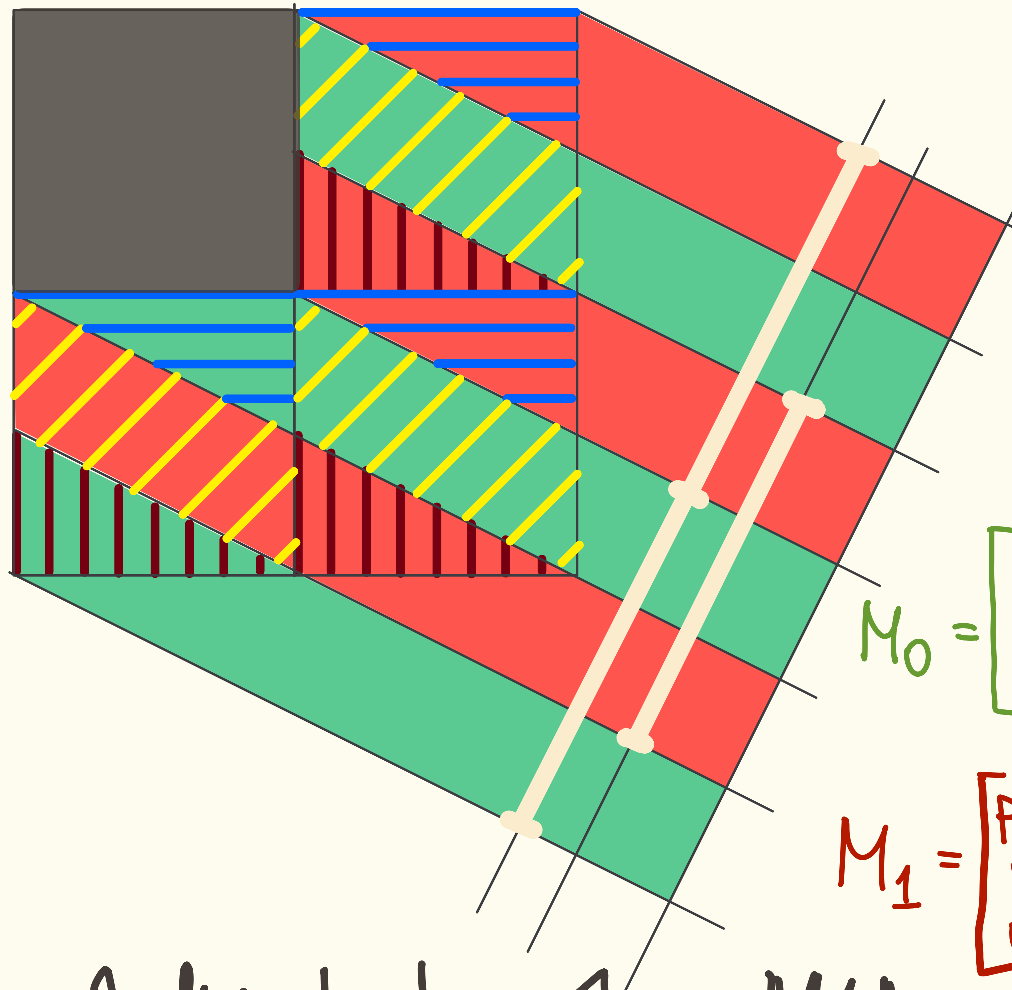


$m_0 = p$, $m_1 = 2p$

$\log m_0 \cdot m_1 > 0 \Leftrightarrow$

$\mathrm{Leb}(\mathrm{proj}\,\Lambda_p) > 0$

branching process in a random environment $\rightarrow$ multitype bpre



$M_0 = \begin{bmatrix} p & 0 & 0 \\ 0 & p & p \\ 0 & p & 0 \end{bmatrix}$, $M_1 = \begin{bmatrix} p & p & 0 \\ p & 0 & p \\ 0 & 0 & p \end{bmatrix}$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{L^n} \cdot \sum_{\underline{i} \in [L]^n} \log \| M_{\underline{i}} \| > 0$$

given: a/ all matrices M_i has a pos. element in all rows & cols.

b/ there is a strictly positive product $M_{i_1} M_{i_2} \dots M_{i_k}$

MBPRE THM [O.-Simon, 2026]

$(\mathcal{Z}_n)_n$ is an MBPRE with

a/ all matrices M_i has a pos. element in all rows & cols.

b/ there is a strictly positive product $M_{i_1} M_{i_2} \dots M_{i_k}$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{L^n} \cdot \sum_{\underline{i} \in [L]^n} \log \|M_{\underline{i}}\| > 0 \iff \lim_{n\to\infty} \mathbb{P}(\tilde{\mathcal{Z}}_n > 0) > 0$$

COROLLARY. [O.-Simon, 2026]

$S = \{ S_i(x) = \frac{x}{L} + t_i \}$, $L \in \mathbb{N}$, $t_i \in \mathbb{R}$.

$M_0, \dots, M_{L-1}$ are the corresponding expectation matrices.

under red conditions on $M_0, \dots, M_{L-1}$:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{L^n} \cdot \sum_{\underline{i} \in [L]^n} \log \|M_{\underline{i}}\| > 0 \iff \mathcal{L}eb(\Lambda_{S,p}) > 0$$

a.s. $\star$

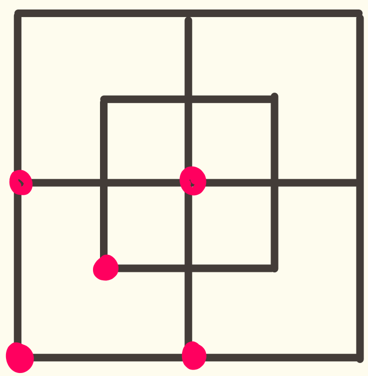
OPEN QUESTIONS.

1) do we need:

a/ all matrices M_i has a pos. element in all rows & cols.,

b/ there is a strictly positive product $M_{i_1} M_{i_2} \dots M_{i_k}$?

Remark. Generally holds for 1-dimensional systems, but not for



(not for this system for example.)

MBPRE THM [O.-Simon, 2026]

$(\mathcal{Z}_n)_n$ is an MBPRE with

a/ all matrices M_i has a pos. element in all rows & cols.

b/ there is a strictly positive product $M_{i_1}M_{i_2}\dots M_{i_k}$

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\cdot\frac{1}{L^n}\cdot \sum_{\underline{i}\in[L]^n} \log\|M_{\underline{i}}\| > 0 \iff \lim_{n\to\infty} \mathbb{P}(\tilde{\mathcal{Z}}_n > 0) > 0$$

COROLLARY. [O.-Simon, 2026]

$S = \{S_i(x) = \frac{x}{L} + t_i\}$, $L\in\mathbb{N}$, $t_i\in\mathbb{R}$.

$M_0,\dots,M_{L-1}$ are the corresponding expectation matrices.

under red conditions on $M_0,\dots,M_{L-1}$:

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\cdot\frac{1}{L^n}\cdot \sum_{\underline{i}\in[L]^n} \log\|M_{\underline{i}}\| > 0 \iff \mathcal{L}eb(\Lambda_{S,p}) > 0$$

a.s. *

OPEN QUESTIONS.

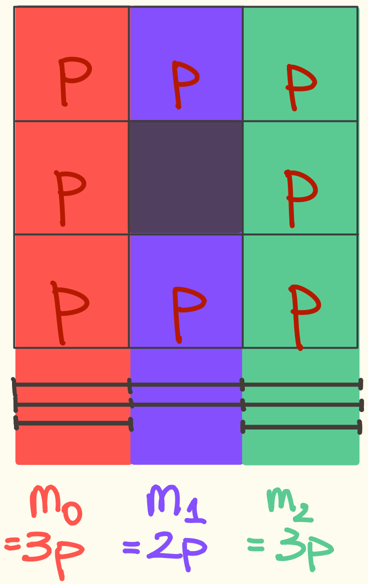
2) Can we broaden this family?

$$S = \{ S_l(x) = 1/2 + t_l i \},\ l \in \mathbb{N},\ t_l \in \mathbb{R}.$$

THANK YOU

FOR YOUR ATTENTION!

EMPTY INTERIOR



Theorem [Falconer-Grimmett]:

if $\exists i \;\; m_i < 1 \Rightarrow \mathrm{Int}(\mathrm{proj}\, \Lambda_p) = \phi$ a.s.

„bad" behavior repeats inside all column.