Properties of a Family of Random Self-Similar Iterated Function Systems on the Line

Orgoványi Vilma
(BME)

Supervisor: Prof. Károly Simon

2026. March 16, Home Defense

PROJECTIONS

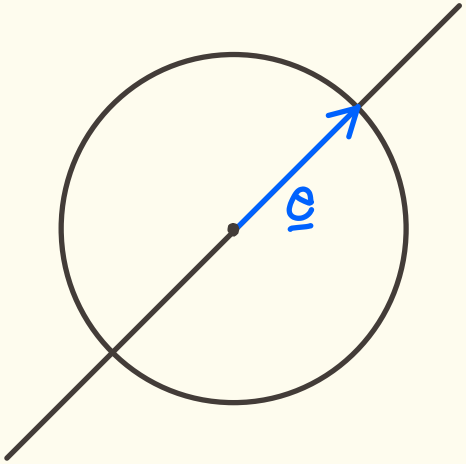
MARSTRAND'S PROJECTION THEOREM

$E \subseteq \mathbb{R}^2$, BOREL SET (1954)

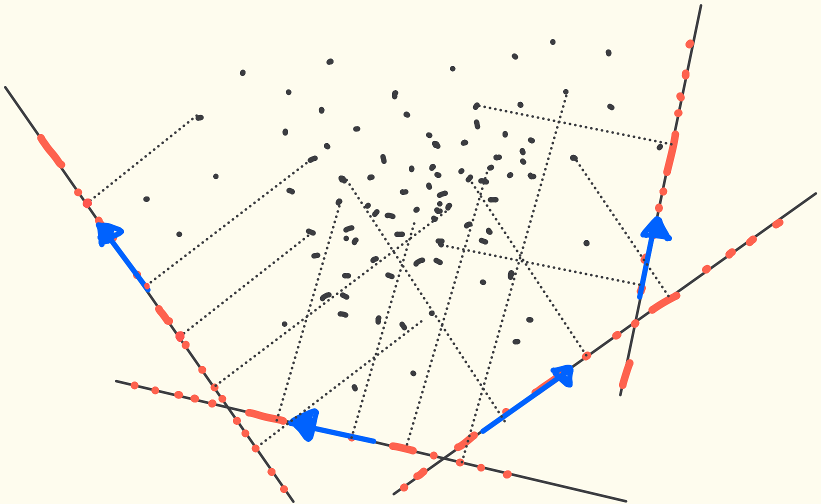
① If $\dim_H E > 1$ then for almost every $\underline{e} \in S^1$ $\mathrm{Leb}(\mathrm{proj}_{\underline{e}} E) > 0$.

② If $\dim_H E \leq 1$ then for almost every $\underline{e} \in S^1$

$$\dim_H(\mathrm{proj}_{\underline{e}} E) = \dim_H E.$$



$\mathrm{proj}_{\underline{e}}$ = orth. projection to line spanned by $\underline{e}$



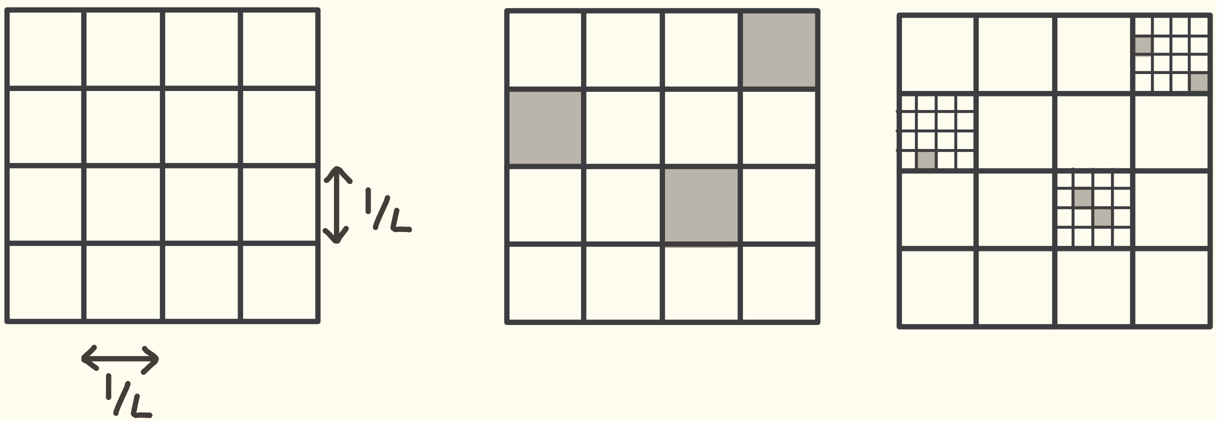
MANDELBROT PERCOLATION

$E \subseteq \mathbb{R}^2$, BOREL SET

① If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.

② If $\dim_H E \leq 1$ then for almost every $e \in S^1$ $\dim_H(\mathrm{proj}_e E) = \dim_H E$.

A/ Construction: Fix $p \in (0,1]$, $L \geq 2$.



$\dots \quad M_p$

$\mathbb{P}(M_p \neq \emptyset) > 0 \iff p > 1/L^2$

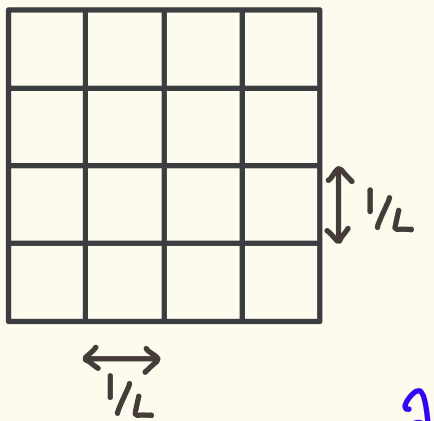
Mandelbrot Percolation

$E \subseteq \mathbb{R}^2$, Borel set

① If $\dim_H E > 1$ then for almost every $e \in S^1$ $\mathrm{Leb}(\mathrm{proj}_e E) > 0$.

② If $\dim_H E \leq 1$ then for almost every $e \in S^1$ $\dim_H(\mathrm{proj}_e E) = \dim_H E$.

A/ Construction: Fix $p \in (0,1]$, $L \geq 2$.



- $\mathbb{P}(M_p \neq \emptyset) > 0 \iff p > 1/L^2$
- $\dim M_p = \dfrac{\log L^2 p}{\log L}$ a.s. conditioned on non-extinction

[Hawkes '81
Falconer '86
Mauldin & Williams '86
Kahane '85]

a.s.* = a.s. conditioned on non-extinction

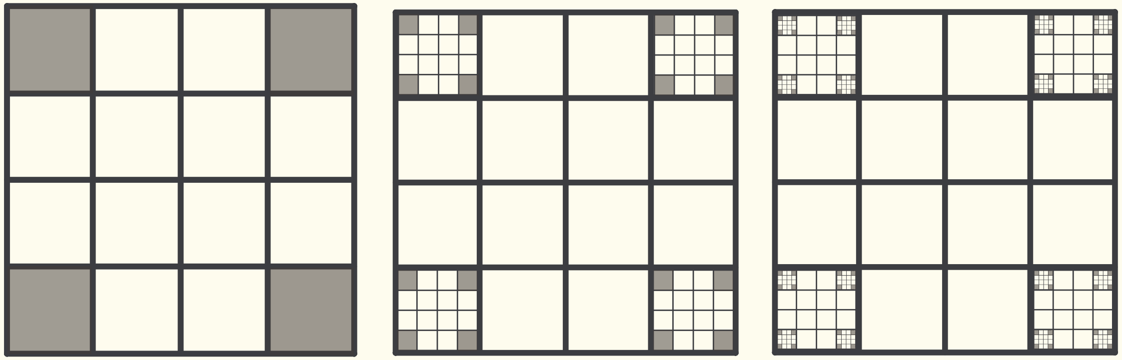
B/ Rams-Simon, '14, '15

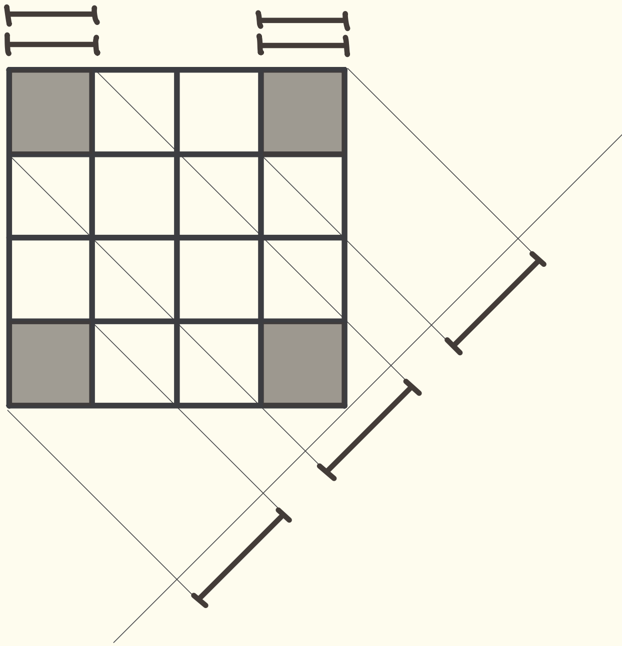
① $\dim_H M_p > 1 \implies \mathrm{Int}(\mathrm{proj}_{\underline{e}} M_p) \neq \emptyset$

② $\dim_H M_p \leq 1 \implies \dim_H(\mathrm{proj}_{\underline{e}} M_p) = \dim_H M_p$

a.s.* for all $\mathrm{proj}_{\underline{e}}$ projections to lines.

EXCEPTIONAL DIRECTIONS: 4-corner set

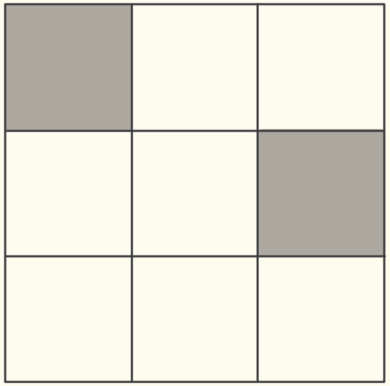




many rational directions are exceptional:

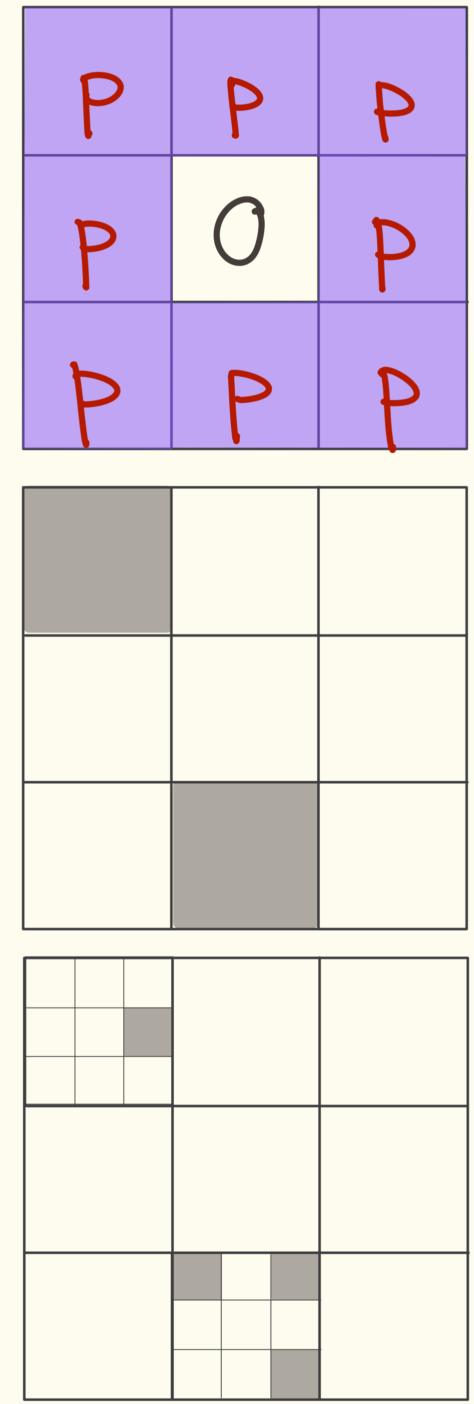
$$\dim(\mathrm{proj}\,\Lambda) < \dim(\Lambda) = 1$$

Generally: $\mathbb{Z}$-grid aligned sets
+ rational projections
can be problematic



EXCEPTIONAL DIRECTIONS: Randomness

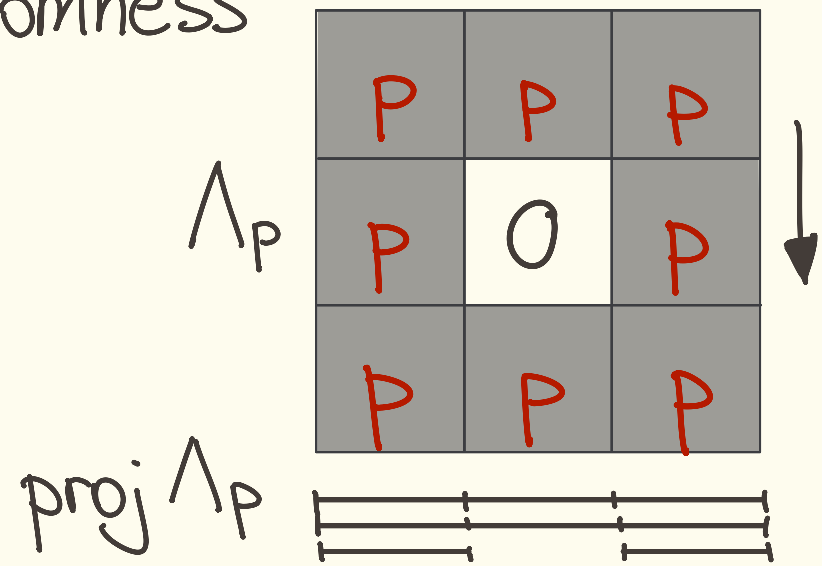
$p \in (0,1]$



$\ldots \Lambda_p$

EXCEPTIONAL DIRECTIONS: Randomness

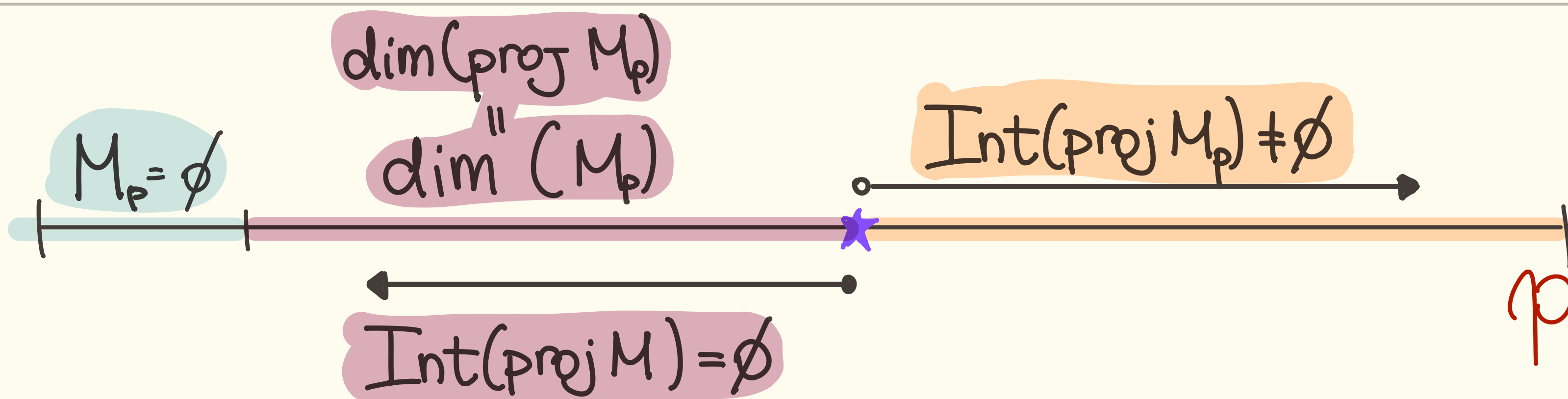
$p \in (0,1]$



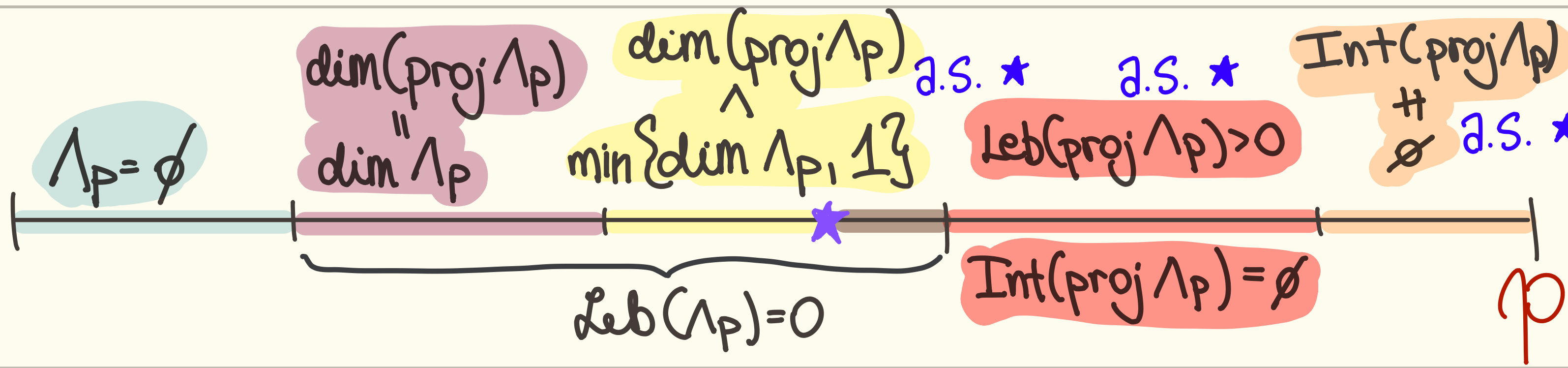
(1) orthogonal projections to the x-axis:

- Dekking-Grimmett '88
- Falconer '89
- Falconer-Grimmet '92

projections of the Mandelbrot percolation:



proj Λ_p :

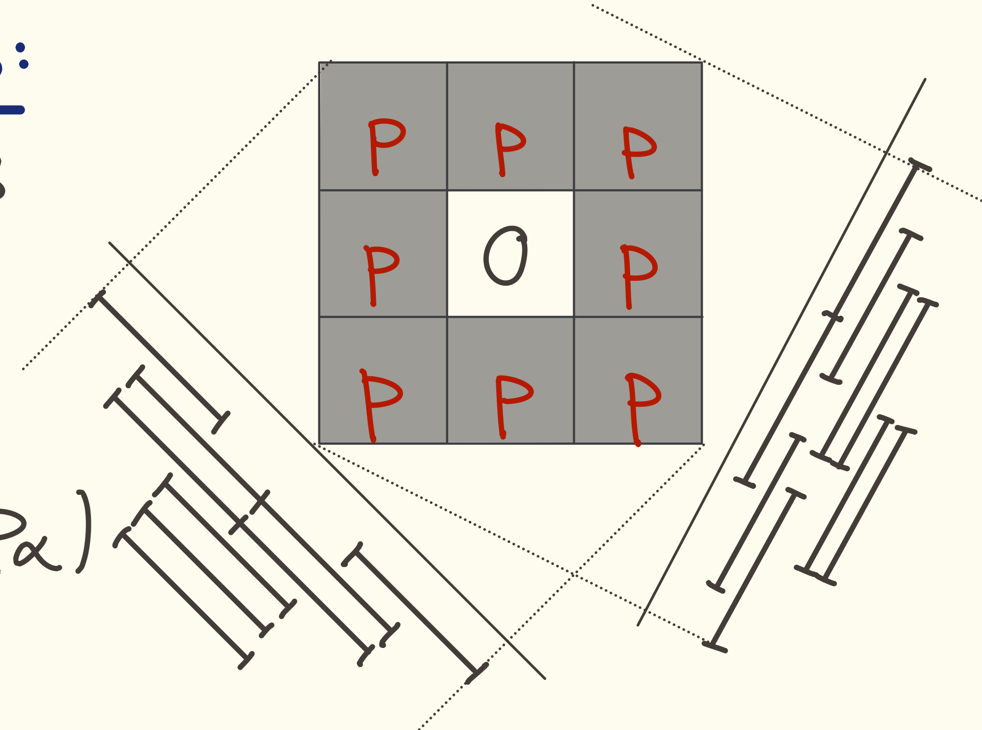


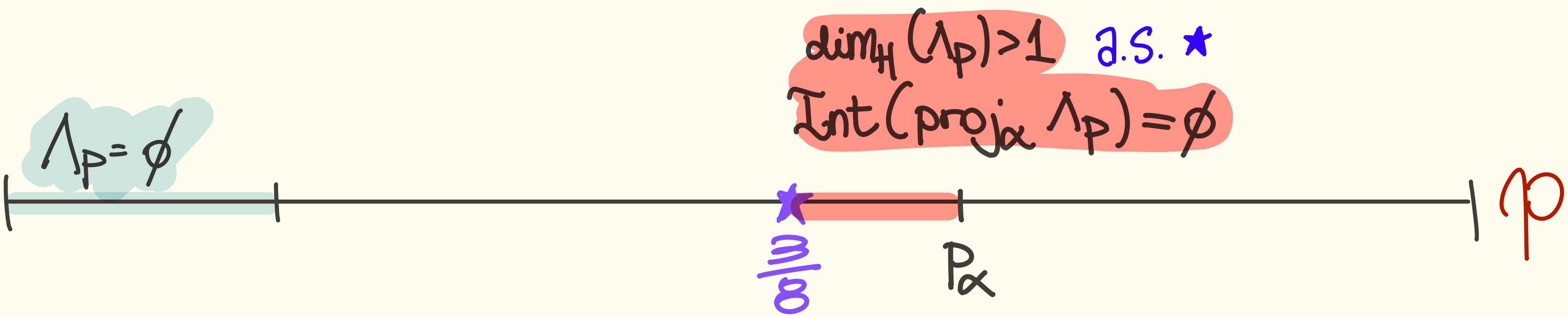
EXCEPTIONAL DIRECTIONS:

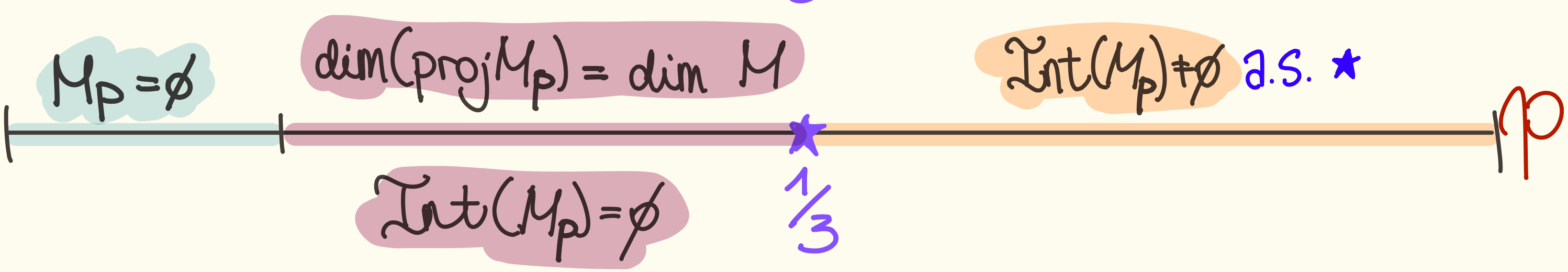
Randomness

(B) Simon-Vágó '18

For any α (rational) there exists a parameter interval $\left(\frac{3}{8}, P_\alpha\right)$







Phase Transitions

Dekking, Falconer, Grimmett:

$p_\emptyset = 1/3$

$\text{Leb}(\Lambda_1(p)) = 0$

$\text{Leb}(\Lambda_1(p)) > 0$ a.s. ★

0.707 — 1

$\text{Int}(\Lambda_1(p)) = \emptyset$

Λ_1

$\text{Leb}(\Lambda_2(p)) > 0$ a.s. ★

$0.6728 \leq \tilde{p} \leq 0.673^*$

$\text{Int}(\Lambda_2(p)) = \emptyset$

Λ_2

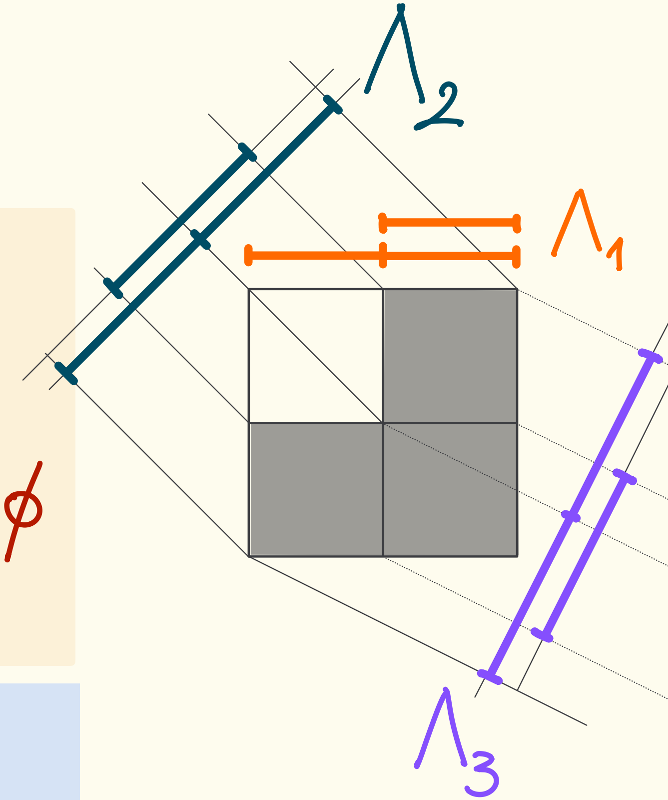
$\text{Int}(\Lambda_3(p)) = \emptyset$

$\text{Int}(\Lambda_3(p)) \neq \emptyset$ a.s. ★

$0.618 \leq p_\star \leq 0.7357$

Λ_3

$\text{Leb}(\Lambda_3(p)) > 0$ & $\text{Int}(\Lambda_3(p)) = \emptyset$ a.s. ★

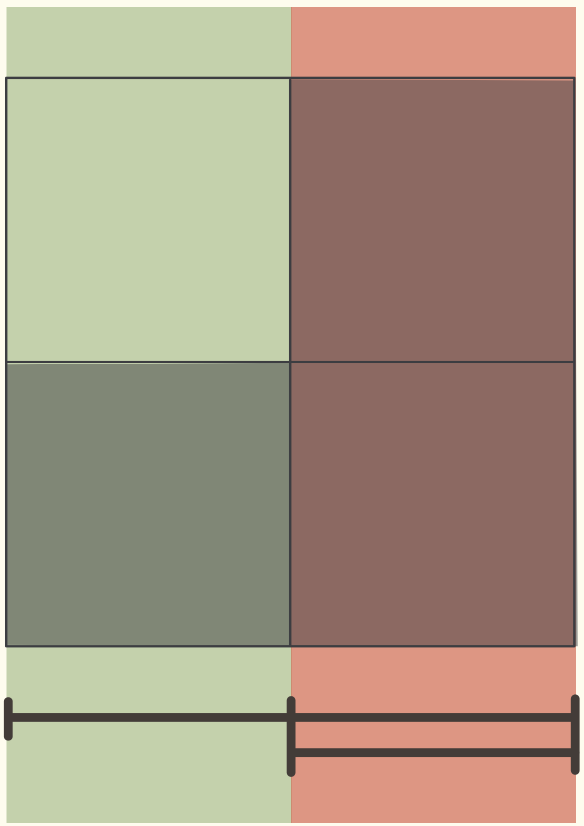


O–Simon

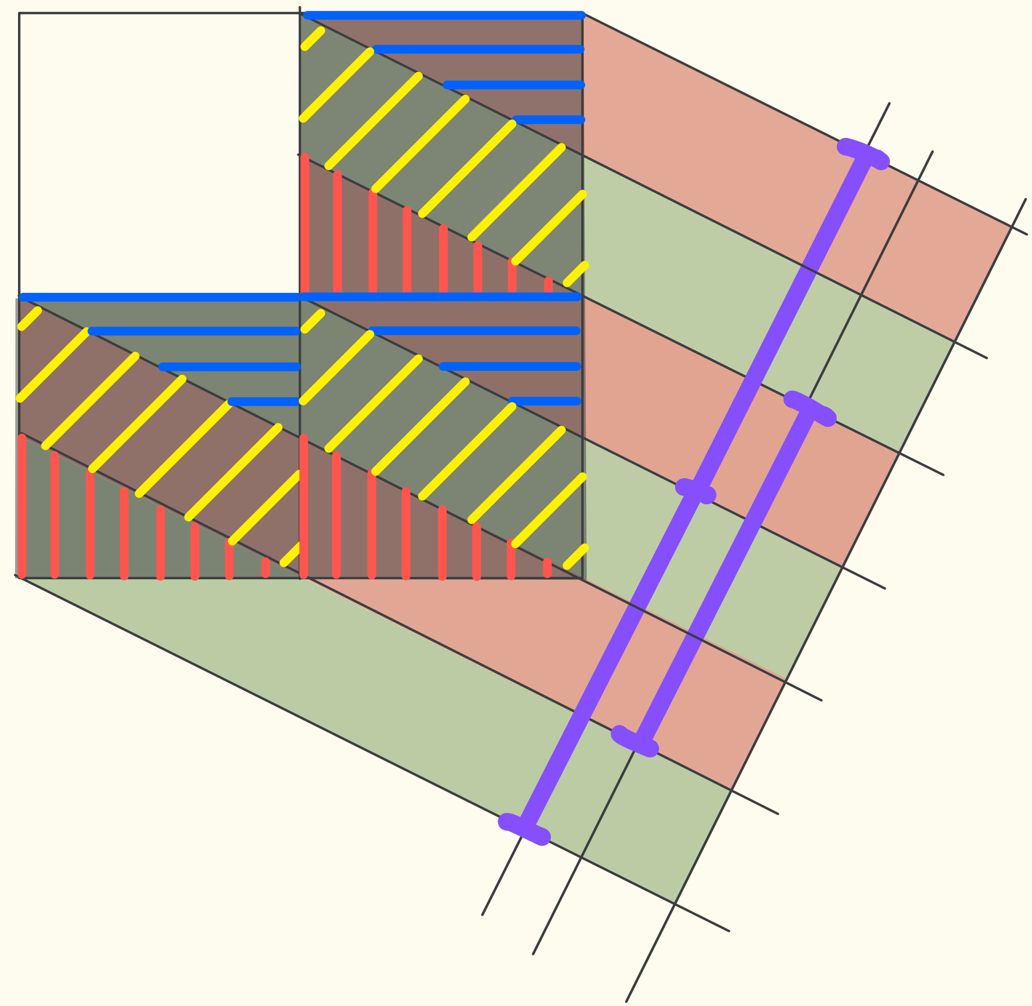
*Pollicott–Vytnova

6

Axis vs. other rational projections

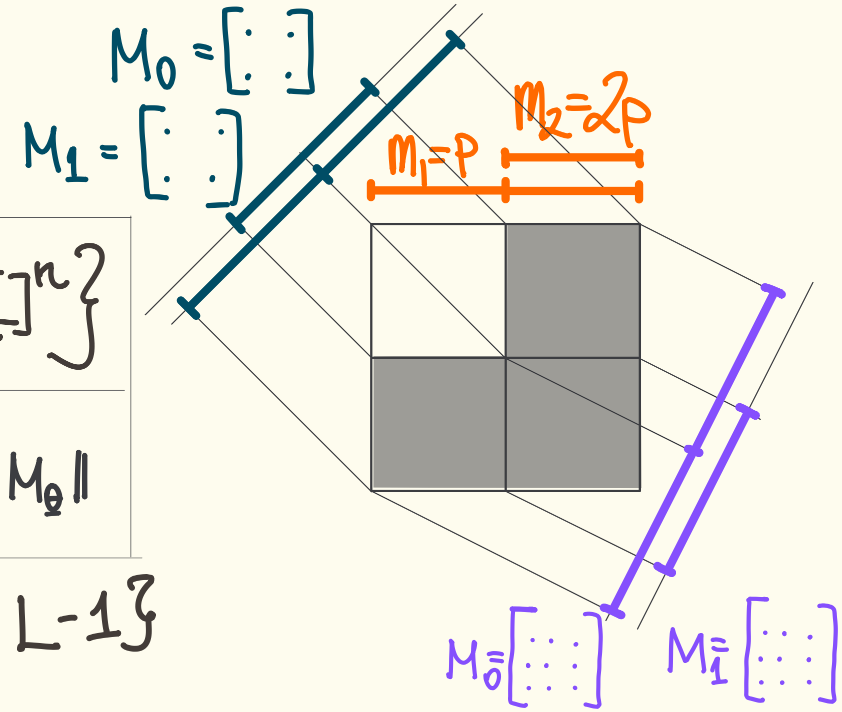


$m_0 = p$, $m_1 = 2p$



$$M_0 = \begin{bmatrix} p & 0 & 0 \\ 0 & p & p \\ 0 & p & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} p & p & 0 \\ p & 0 & p \\ 0 & 0 & p \end{bmatrix}$$

Phase Transitions



$\check{s}(p)$	$\log \min\limits_{\theta\in[L]} m_\theta$	$\lim\limits_{n\to\infty} \min\left\{\frac{1}{n}\cdot\log\lVert M_{\underline{\theta}}\rVert : \underline{\theta}\in[L]^n\right\}$
$\lambda(p)$	$\frac{1}{L}\sum\limits_{\theta\in[L]}\log m_\theta$	$\lim\limits_{n\to\infty} \frac{1}{n}\,\frac{1}{L^n}\cdot\sum\limits_{\underline{\theta}\in[L]^n}\log\lVert M_{\underline{\theta}}\rVert$

$[L]=\{0,\dots,L-1\}$

Dekking, Falconer, Grimmett

$p_\phi = 1/3$ 1

$\mathrm{Leb}(\Lambda_1(p))=0$ $\mathrm{Leb}(\Lambda_1(p))>0$

0.707

$\mathrm{Int}(\Lambda_1(p))=\phi$

- $\mathrm{Int}(\Lambda_1(p))\neq\phi$ iff $\check{s}(p)>0$
- $\mathrm{Leb}(\Lambda_1(p))>0$ iff $\lambda(p)>0$

$\mathrm{Leb}(\Lambda_2(p))=0$ $\mathrm{Leb}(\Lambda_2(p))>0$

$0.6728\leq\tilde{p}\leq 0.673$

$\mathrm{Int}(\Lambda_2(p))=\phi$

- $\check{s}(p)<0 \Rightarrow \mathrm{Int}(\Lambda_2(p))=\phi$
- $\mathrm{Leb}(\Lambda_2(p))>0$ iff $\lambda(p)>0$

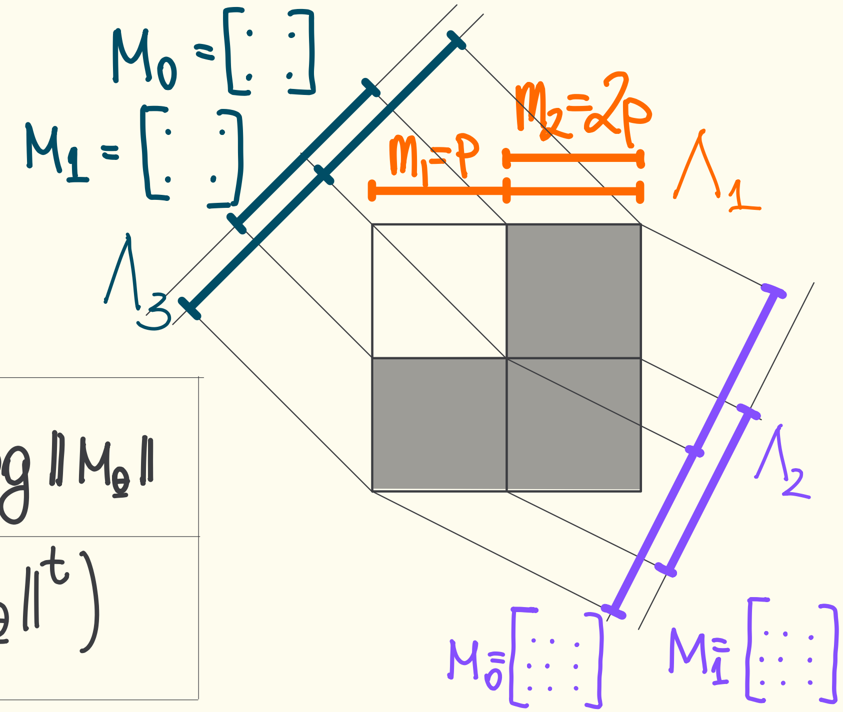
$\mathrm{Int}(\Lambda_3(p))=\phi$ $\mathrm{Int}(\Lambda_3(p))\neq\phi$

$0.618\leq p_\star \leq 0.7357$

$\mathrm{Leb}(\Lambda_3(p))>0$ & $\mathrm{Int}(\Lambda_3(p))=\phi$

O-Simon

Phase Transitions



$\lambda(p)$	$\frac{1}{L}\sum_{\theta\in[L]}\log m_\theta$	$\lim_{n\to\infty}\frac{1}{n}\frac{1}{L^n}\sum_{\underline{\theta}\in[L]^n}\log\lVert M_{\underline{\theta}}\rVert$
$P_p(t)$	$\log\sum_{\theta\in[L]} m_\theta^t$	$\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\underline{\theta}\in[L]^n}\lVert M_{\underline{\theta}}\rVert^t\right)$

$[L]=\{0,\dots,L-1\}$

$$\dim_H \Lambda_i(p)=\begin{cases}1, & \text{if } \lambda(p)>0\\ \inf_t \dfrac{P(t)}{\log(L)}, & \text{if } \lambda(p)\le 0,\ P'(1)\ge 0\\ \dfrac{\log(Mp)}{\log(L)}, & \text{if } P'(1)<0\end{cases}$$

$i=1,2,3$

(For Λ_1: Dekking & Grimmett '88, Falconer '89
Λ_2, Λ_3: O-Simon)

Theorem A

IFS on the line

$$\mathcal{S}=\left\{S_i(x)=\tfrac{1}{L}x+t_i\right\}_{i=1}^{M},$$

$t_i\in\mathbb{Q}$, $t_1\le\ldots\le t_M$

Percolation on its cylinder sets

$$\left(\left\{S_{\underline{i}}\left(\left[0,t_M\cdot\tfrac{L}{L-1}\right]\right)\right\}_{\underline{i}\in\{1,\ldots,M\}^n}\right)$$

$$P(t)=\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\underline{\theta}}\|M_{\underline{\theta}}\|^t\right)$$

$$\lambda=\lim_{n\to\infty}\frac{1}{n}\frac{1}{L^n}\cdot\sum_{\underline{\theta}}\log\|M_{\underline{\theta}}\|$$

Λ_p = random attractor on the line

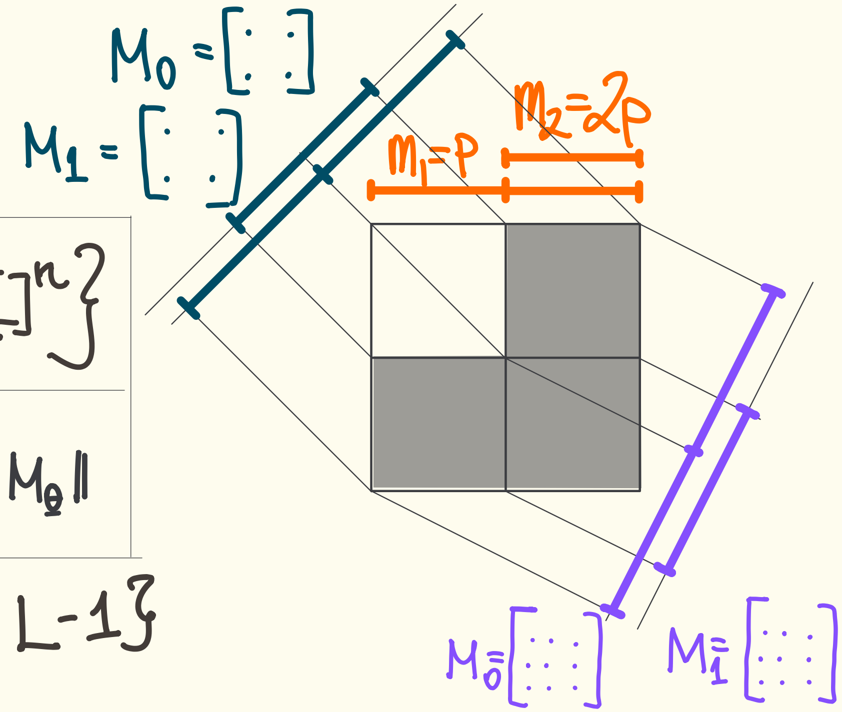
$\{M_\theta\}_{\theta=0}^{L-1}$: expectation matrices

Dimension formula (O.-Simon, '26+)

$M_0,\ldots,M_{L-1}$ are allowable, having a positive product

$$\dim\Lambda_p=\begin{cases}1, & \text{if } \lambda>0\\ \inf\limits_{t\in[0,1]}\dfrac{P(t)}{\log(L)}, & \text{if } \lambda\le 0 \;\&\; P'(1)>0\\ \dim_S\Lambda=\dfrac{\log Mp}{\log L}, & \text{if } P'(1)\le 0\end{cases}$$

Phase Transitions



$\check{s}(p)$	$\log \min\limits_{\theta\in[L]} m_\theta$	$\lim\limits_{n\to\infty} \min\{\frac{1}{n}\cdot\log\lVert M_{\underline{\theta}}\rVert : \underline{\theta}\in[L]^n\}$
$\lambda(p)$	$\frac{1}{L}\sum\limits_{\theta\in[L]}\log m_\theta$	$\lim\limits_{n\to\infty} \frac{1}{n}\frac{1}{L^n}\cdot\sum\limits_{\underline{\theta}\in[L]^n}\log\lVert M_{\underline{\theta}}\rVert$

$[L]=\{0,\dots,L-1\}$

Dekking, Falconer, Grimmett

$p_\emptyset = 1/3$ 1

$\mathrm{Leb}(\Lambda_1(p))=0$ $\mathrm{Leb}(\Lambda_1(p))>0$

0.707

$\mathrm{Int}(\Lambda_1(p))=\emptyset$

- $\mathrm{Int}(\Lambda_1(p))\neq\emptyset$ iff $\check{s}(p)>0$
- $\mathrm{Leb}(\Lambda_1(p))>0$ iff $\lambda(p)>0$

$\mathrm{Leb}(\Lambda_2(p))=0$ $\mathrm{Leb}(\Lambda_2(p))>0$

$0.6728\leq\tilde{p}\leq 0.673$

$\mathrm{Int}(\Lambda_2(p))=\emptyset$

- $\check{s}(p)<0 \Rightarrow \mathrm{Int}(\Lambda_2(p))=\emptyset$
- $\mathrm{Leb}(\Lambda_2(p))>0$ iff $\lambda(p)>0$

$\mathrm{Int}(\Lambda_3(p))=\emptyset$ $\mathrm{Int}(\Lambda_3(p))\neq\emptyset$

$0.618\leq p_\star\leq 0.7357$

$\mathrm{Leb}(\Lambda_3(p))>0$ & $\mathrm{Int}(\Lambda_3(p))=\emptyset$

O-Simon

Theorem B

IFS on the line

$\mathcal{S} = \{ S_i(x) = \frac{1}{L} x + t_i \}_{i=1}^{M}$,

$t_i \in \mathbb{Q}$, $t_1 \leq \ldots \leq t_M$

percolation on its cylinder sets

$\left(\{ S_{\underline{i}}([0, t_M \cdot \frac{L}{L-1}]) \}_{\underline{i} \in \{1,\ldots,M\}^n} \right)$

$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\underline{\theta}} \|M_{\underline{\theta}}\|^t \right)$

$\lambda = \lim_{n\to\infty} \frac{1}{n} \frac{1}{L^n} \cdot \sum_{\underline{\theta}} \log \|M_{\underline{\theta}}\|$

Λ_p = random attractor on the line

$\{M_\theta\}_{\theta=0}^{L-1}$: expectation matrices

Positivity of Lebesgue measure (O.-Simon, '26₁)

$M_0, \ldots, M_{L-1}$ are allowable, having a positive product

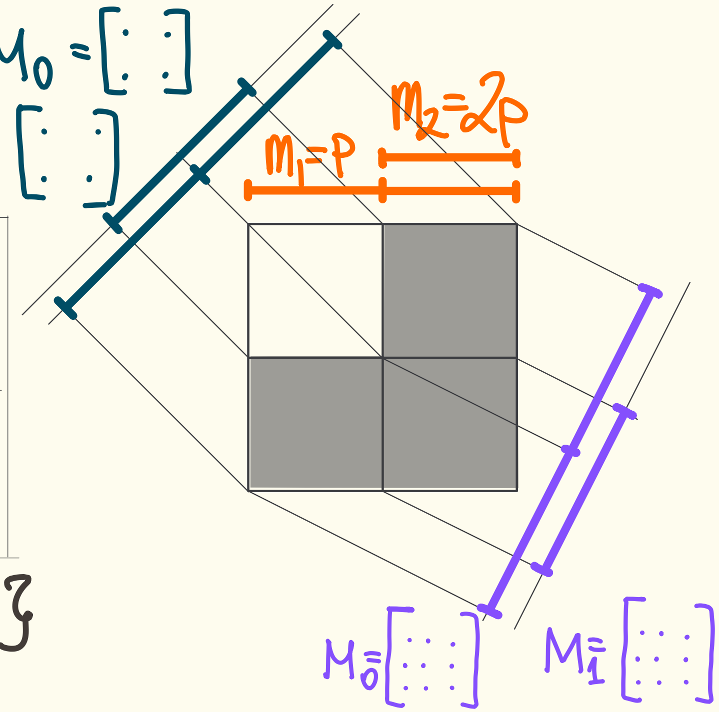
$\mathcal{L}eb(\Lambda) > 0$ a.s. conditioned on non-extinction

iff

$\lambda > 0$

Based on a theorem about the survival of Multitype branching processes in random environments. (O.-Simon, '26₂)

PHASE TRANSITIONS



$\check{s}(p)$	$\log \min_{\theta\in[L]} m_\theta$	$\lim_{n\to\infty} \min\left\{\frac{1}{n}\cdot\log\lVert M_{\underline{\theta}}\rVert : \underline{\theta}\in[L]^n\right\}$
$\lambda(p)$	$\frac{1}{L}\sum_{\theta\in[L]}\log m_\theta$	$\lim_{n\to\infty}\frac{1}{n}\frac{1}{L^n}\cdot\sum_{\underline{\theta}\in[L]^n}\log\lVert M_{\underline{\theta}}\rVert$

$[L]=\{0,\dots,L-1\}$

Dekking, Falconer, Grimmett

$p_\emptyset = 1/3$

$Leb(\Lambda_1(p))=0$ $\quad$ $Leb(\Lambda_1(p))>0$ $\quad$ 1

0.707

$Int(\Lambda_1(p))=\emptyset$

- $Int(\Lambda_1(p))\neq\emptyset$ iff $\check{s}(p)>0$
- $Leb(\Lambda_1(p))>0$ iff $\lambda(p)>0$

$Leb(\Lambda_2(p))=0$ $\quad$ $Leb(\Lambda_2(p))>0$

$0.6728\leq\tilde{p}\leq 0.673$

$Int(\Lambda_2(p))=\emptyset$

- $\check{s}(p)<0 \Rightarrow Int(\Lambda_2(p))=\emptyset$
- $Leb(\Lambda_2(p))>0$ iff $\lambda(p)>0$

$Int(\Lambda_3(p))=\emptyset$ $\quad$ $Int(\Lambda_3(p))\neq\emptyset$

$0.618\leq p_\star\leq 0.7357$

$Leb(\Lambda_3(p))>0$ & $Int(\Lambda_3(p))=\emptyset$

O-Simon

Theorem C

IFS on the line

$\mathcal{S} = \{ S_i(x) = \frac{1}{L} x + t_i \}_{i=1}^{M}$,

$t_i \in \mathbb{Q}$, $t_1 \leq \dots \leq t_M$

percolation on its cylinder sets

$(\{ S_{\underline{i}}([0, t_M \cdot \frac{L}{L-1}]) \}_{\underline{i} \in \{1,\dots,M\}^n})$

$\check{s} = \lim_{n \to \infty} \min \{ \frac{1}{n} \cdot \log \| M_{\underline{\theta}} \| : \underline{\theta} \in [L]^n \}$

Λ_p = random attractor on the line

$\{ M_\theta \}_{\theta=0}^{L-1}$: expectation matrices

Existence of interior points (O.-Simon, 2G1)

$M_0, \dots, M_{L-1}$ allowable:

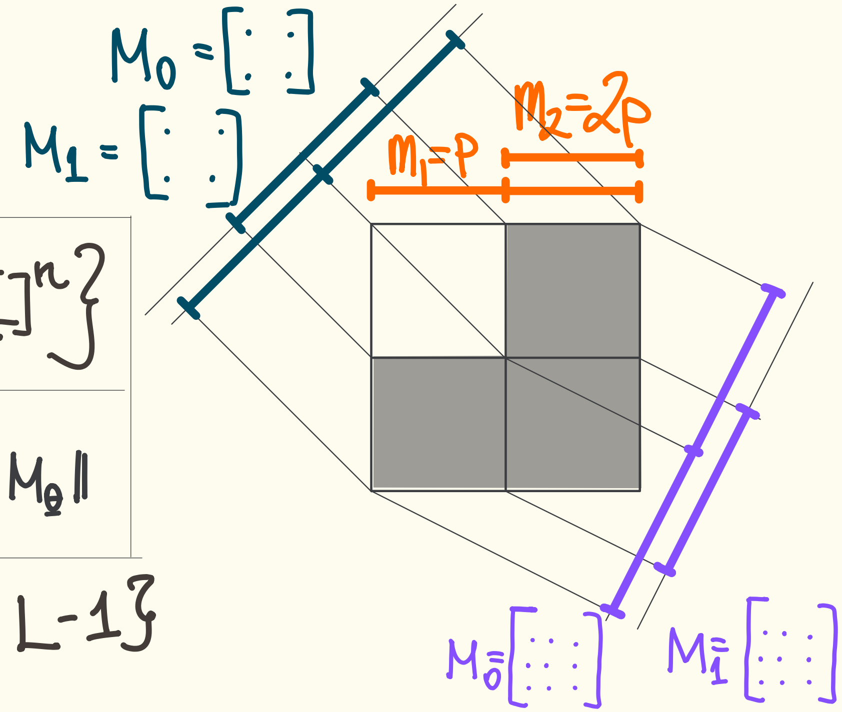
A/ If $\check{s} < 0 \Rightarrow \mathrm{Int}(\Lambda) = \emptyset$ a.s.

B/ If $\exists\, \mathcal{U} = \{ \underline{u}_1, \dots, \underline{u}_m \}$, $\underline{u}_i = (u_i(1), \dots u_i(N))$ $(u_i(k) \in \mathbb{N})$ s.t.

1. $\exists\, u^* \in \{1,\dots,N\}$, $\underline{\theta}^* \in \{0,\dots,L-1\}^{n^*}$, $\underline{v}^* \in \mathcal{U}$: $e_{u^*}^T M_{\underline{\theta}^*} \geq \underline{v}^*$
2. $\exists\, \gamma > 1$, $n \in \mathbb{N}$, $\forall \underline{u} \in \mathcal{U}$, $\underline{\theta} \in \{0,\dots,L-1\}^n$ $\exists\, \underline{v} \in \mathcal{U}$: $\underline{u}^T M_{\underline{\theta}} \geq \gamma \underline{v}$

then $\mathrm{Int}(\Lambda) \neq \emptyset$ a.s. conditioned on non-extinction.

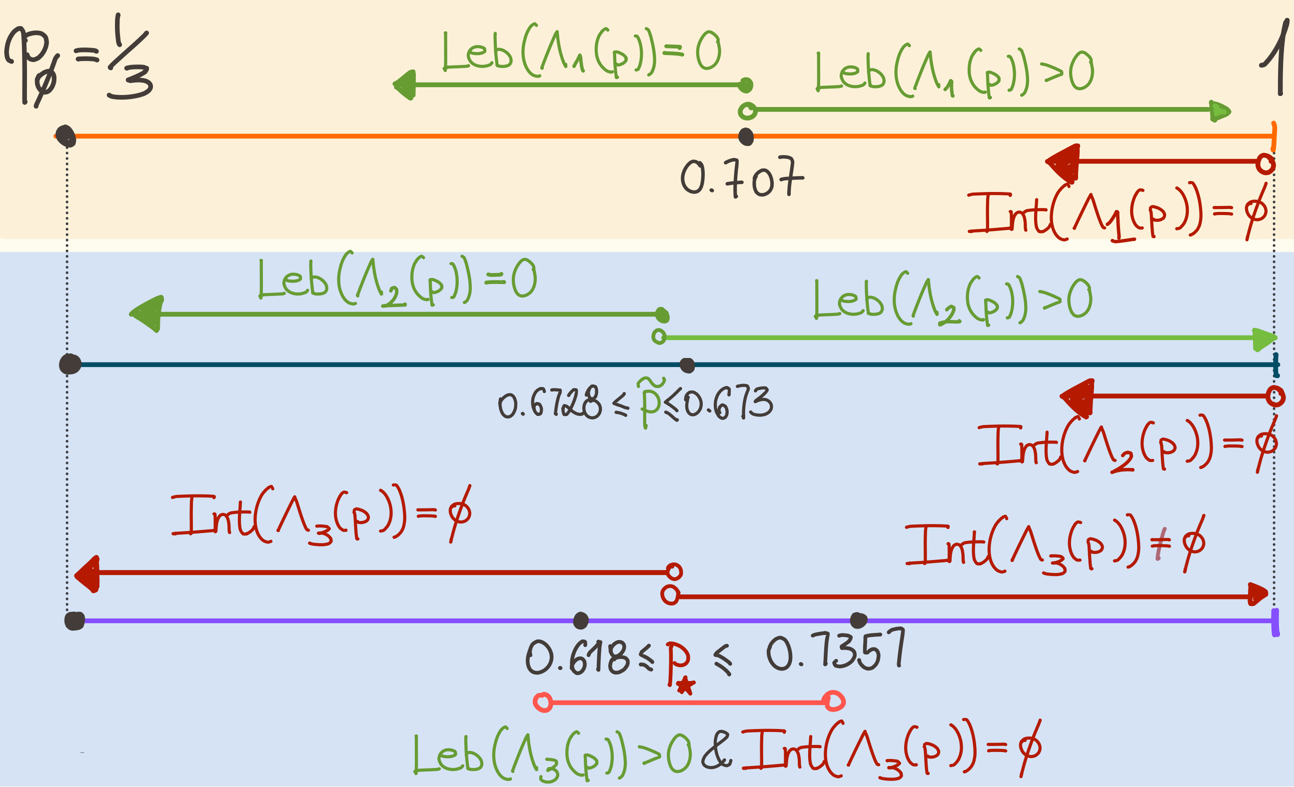
PHASE TRANSITIONS



$\check{s}(p)$	$\log \min_{\theta\in[L]} m_\theta$	$\lim_{n\to\infty} \min\{\frac{1}{n}\cdot\log\|M_{\underline{\theta}}\| : \underline{\theta}\in[L]^n\}$
$\lambda(p)$	$\frac{1}{L}\sum_{\theta\in[L]}\log m_\theta$	$\lim_{n\to\infty}\frac{1}{n}\,\frac{1}{L^n}\cdot\sum_{\underline{\theta}\in[L]^n}\log\|M_{\underline{\theta}}\|$

$[L]=\{0,\dots,L-1\}$

Dekking, Falconer, Grimmett



- $Int(\Lambda_1(p))\neq\phi$ iff $\check{s}(p)>0$
- $Leb(\Lambda_1(p))>0$ iff $\lambda(p)>0$

- $\check{s}(p)<0 \Rightarrow Int(\Lambda_2(p))=\phi$
- $Leb(\Lambda_2(p))>0$ iff $\lambda(p)>0$

Ö-Simon

Theorem D

IFS on the line

$\mathcal{S} = \{ S_i(x) = \frac{1}{\ell} x + t_i \}_{i=1}^{M}$,

$t_i \in \mathbb{Q}$, $t_1 \leq \ldots \leq t_M$

percolation on its cylinder sets

$(\{ S_{\underline{i}}([0, t_M \cdot \frac{\ell}{\ell - 1}]) \}_{\underline{i} \in \{1,\ldots,M\}^n})$

$P(t) = \lim_{n \to \infty} \frac{1}{n} \log (\sum \| M_{\underline{\theta}} \|^t)$

$\lambda = \lim_{n \to \infty} \frac{1}{n} \frac{1}{\ell^n} \cdot \sum \log \| M_{\underline{\theta}} \|$

$\check{S} = \lim_{n \to \infty} \min \{ \frac{1}{n} \cdot \log \| M_{\underline{\theta}} \| : \underline{\theta} \in [L]^n \}$

Λ_p = random attractor on the line

$\{ M_\theta \}_{\theta=0}^{L-1}$: expectation matrices

Existence of an interval of probabilities where Λ_p has positive Lebesgue measure but empty interior (D-S, 26)

If $M_0, \ldots, M_{L-1}$ allowable, has positive product and $P(t)$ is strictly convex on a subinterval of $(-\infty, 0)$ then $\check{S} < \lambda$ and for $p \in (e^{-\lambda}, e^{-\check{S}})$ the random attractor has positive Lebesgue measure but empty interior almost surely conditioned on non-extinction.

Thank you for your attention!

The papers the dissertation is based on:

1. Interior points and Lebesgue measure of overlapping Mandelbrot percolation sets.
   (joint with Károly Simon).
   To appear in Stochastic Processes and their Applications
2. Multitype branching processes in random environments with not strictly positive expectation matrices.
   (joint with Károly Simon)
   Bernoulli (2026)
3. Projections of the random Menger sponge.
   (joint with Károly Simon)
   The Asian Journal of Mathematics (2024)

3+. Dimensions of some statistically self-similar sets.
   (joint with Károly Simon)
   under preparation.

Bibliography

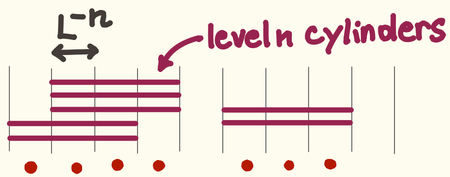
- Marstrand '54 – J.M. Marstrand. Some fundamental geometrical properties of plane sets of fractional dimensions. Proc. Lond. Math. Soc. 1954.
- Hawkes '81 – J. Hawkes. Trees generated by a simple branching process. J. Lond. Math. Soc. 1981
- Kahane '85 – J.P. Kahane. On multiplicative chaos. Ann. Sci. Math Québec. 1985.
- Falconer '86 - K.J. Falconer. Random fractals. Math. Proc. Cambridge Philos. Soc. 1986.
- Mauldin- Williams '86 – R.D. Mauldin- S.C. Williams. Random recursive constructions: Asymptotic and geometric and topological properties. Trans. Amer. Math. Soc. 1986.
- Rams- Simon '14 – M. Rams – K. Simon. The dimension of projections of fractal percolations. J. Stat. Phis. 2014.
- Rams- Simon '14 – M. Rams – K. Simon. Projections of fractal percolations. Ergod. Theory Dyn. Syst. 2015.
- Dekking- Grimmett '88 – F.M. Dekking- G.R. Grimmett. Superbranching processes and projections of random Cantor sets. Probab. Theory Related Fields. 1988.
- Falconer '89 – K.J. Falconer. Projections of random Cantor sets. J. Theoret. Probab. 1989.
- Falconer – Grimmett '92 – K.J. Falconer- G.R. Grimmett. On the geometry of random Cantor sets and Fractal percolation. J. Theoret. Probab. 1992.
- Simon- Vágó '18 – K. Simon - L. Vágó. Fractal percolations. Banach Center Publications. 2018.

THM. A (PROOF IDEA)

$M_0, \dots, M_{L-1}$ are allowable, having a positive product &

$$\dim \Lambda_{\vec{p}} = \begin{cases} 1, & \text{if } \lambda > 0 \\ \inf_{t \in [0,1]} \frac{P(t)}{\log(L)}, & \text{if } \lambda \leq 0 \ \& \ P'(1) > 0 \\ \dim_S \Lambda = \frac{\log M_p}{\log L}, & \text{if } P'(1) \leq 0 \end{cases}$$

① upper bound



$Y_n = \#\{L^{-n}$ grid elements intersecting the Int of a cylinder$\}$

$$\dim_H \Lambda \overset{(=)}{\leq} \overline{\dim}_B \Lambda = \limsup_{n \to \infty} \frac{\log Y_n}{n \log L}$$

i) $\exists \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(Y_n)$

[~ subadditive seq. + Fekete's subadditive lemma]

ii) $\limsup_{n \to \infty} \frac{\log Y_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(Y_n)$

[Markov + Borel-Cantelli]

iii) $\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(Y_n) \leq \inf_{t \in [0,1]} P(t)$

[Linearity of expectation + Jensen + Markov]

THM. A (PROOF IDEA)

$M_0, \dots, M_{L-1}$ are allowable, having a positive product $\mathcal{L}$

$$\dim \Lambda_{\bar{p}} = \begin{cases} 1, & \text{if } \lambda > 0 \\ \inf_{t \in [0,1]} \frac{P(t)}{\log(L)}, & \text{if } \lambda \le 0 \ \& \ P'(1) > 0 \\ \dim_S \Lambda = \frac{\log M_P}{\log L}, & \text{if } P'(1) \le 0 \end{cases}$$

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum \|M_{\underline{\theta}}\|^t\right)$$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \frac{1}{\mathcal{L}^n} \cdot \sum \log \|M_{\underline{\theta}}\|$$

1. lower bound

3+2 scenarios

$\boxed{\text{I.}}$ $\lambda > 0$: Pos. Leb. m $\Rightarrow$ dim = 1 ✓

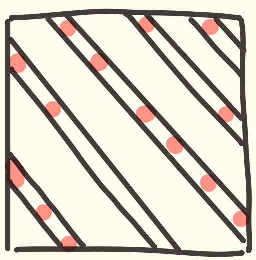
$\boxed{\text{II.}}$ $\lambda < 0$, $P'(1) > 0$:

Thm of Feng-Lau: $\forall \varepsilon > 0$ $\exists$ (deterministic) μ_t ergodic measure

a) $\dim_H \mu_t \ge P(t) - \varepsilon$

Thm B modified: b) $\mu_t(\Lambda) > 0$ a.s. cond. on non-extinction.

$\boxed{\text{III.}}$ $P'(1) < 0$ (Rough idea.)



a) if we delete „big" slices strategically the dim remains the same

b) projection of a set consisting of small slices does not reduce the dimension

THM. B (PROOF IDEA)

$M_0, \dots, M_{L-1}$ are allowable, having a positive product

$Leb(\Lambda) > 0$ a.s. conditioned on non-extinction

iff $\lambda > 0$

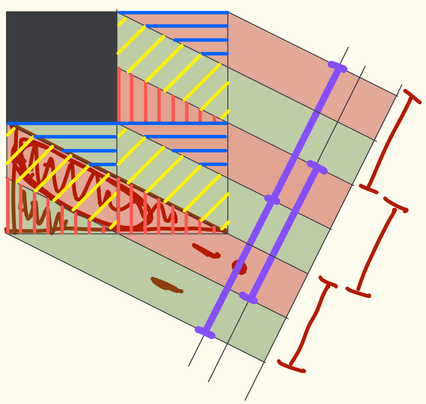
$$\lambda = \lim_{n \to \infty} \frac{1}{n} \frac{1}{L^n} \sum \log \| M_\theta \|$$

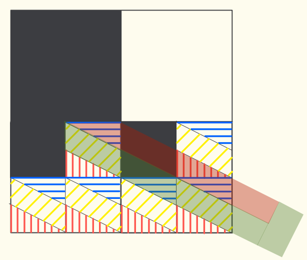
① Enough: There is a set K:

- $Leb(K) > 0$
- $\forall x \in K : \mathbb{P}(x \in \Lambda) > 0$

② Multitype branching process in varying & random environment

Based on a theorem about the survival of Multitype branching processes in random environments. (O.-Simon, '25)





$$M_0 = \begin{bmatrix} p & 0 & 0 \\ 0 & p & p \\ 0 & p & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} p & p & 0 \\ p & 0 & p \\ 0 & 0 & p \end{bmatrix}$$

- Existing results: Athreya-Karlin '71
  - Kaplan '74
  - Tanny '81

↳ strong positivity assumptions

(reason: λ is def. with the greatest column sum, but the smallest column sum describes the system. They need to be approx. equal.)

↳ '97 Hennion

THM. C

$M_0,\dots,M_{L-1}$ allowable:

A/ If $\check{s}<0 \Rightarrow \mathrm{Int}(\Lambda)=\emptyset$ a.s.

B/ $\exists\, \mathcal{U}=\{u_1,\dots,u_m\}$, $u_i=(u_i(1),\dots,u_i(N))$ $(u_i(k)\in\mathbb{N})$ s.t.

① $\exists\, u^x\in\mathcal{U},\ \theta^x\in\{0,\dots,L-1\}^{n^x},\ v^x\in\{1,\dots,N\}$: $e_{u^x}^T M_{\theta^x}\geq v^x$

② $\exists\, \gamma>1,\ n\in\mathbb{N},\ \forall u\in\mathcal{U},\ \theta\in\{0,\dots,L-1\}^n\ \exists\, v\in\mathcal{U}$: $u^T M_\theta\geq\gamma v$

$$\check{s}=\lim_{n\to\infty}\min\left\{\frac{1}{n}\cdot\log\|M_\theta\| : \theta\in[L]^n\right\}$$

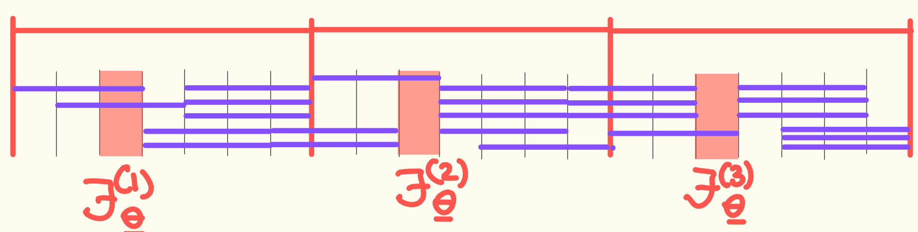
A/ $\check{s}<0 \Rightarrow \exists\, \theta=\theta_1\dots\theta_n\in\{0,\dots,L-1\}^n$

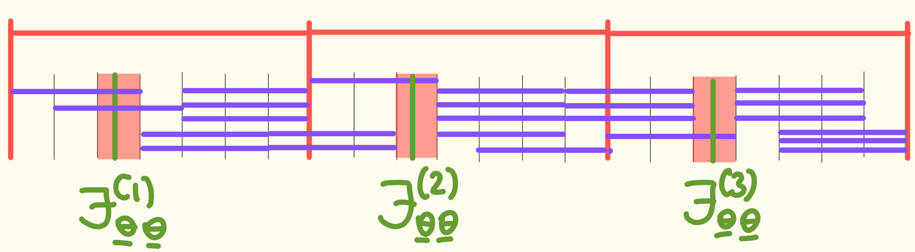
s.t. $\|M_\theta\|=\varepsilon<1$

$\|M_{\theta\theta\dots\theta}\|\leq\varepsilon^n\to 0$

the points $\bigcup_{u=1}^{N}\bigcup_{k=1}^{\infty}\bigcup_{C_k\in[L]^k}\bigcap_{n=1}^{\infty}\mathcal{J}^{(u)}_{C_k\theta^n}$ are not contained in the attractor [markov]

each iv contains an iv of the form: $\bigcup_{u=1}^{N}\bigcup_{k=1}^{\infty}\bigcup_{C_k\in[L]^k}\mathcal{J}^{(u)}_{C_k}$





$M_0,\dots,M_{L-1}$ allowable:

A/ If $\check{s}<0 \Rightarrow \mathrm{Int}(\Lambda)=\emptyset$ a.s.

B/ $\exists\, \mathcal{U}=\{u_1,\dots,u_m\}$, $u_i=(u_i(1),\dots,u_i(N))$ $(u_i(k)\in\mathbb{N})$ s.t.

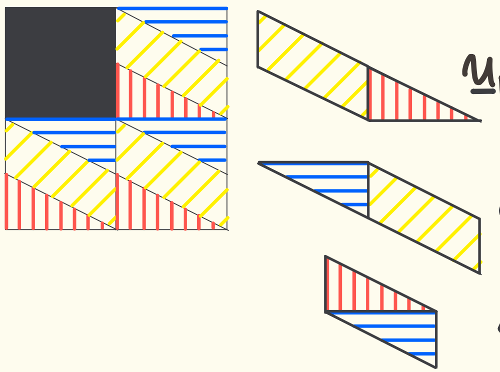
① $\exists\, u^x\in\mathcal{U},\ \theta^x\in\{0,\dots,L-1\}^{n^x},\ v^x\in\{1,\dots,N\}:\ e_{u^x}^T M_{\theta^x}\geq v^x$

② $\exists\, \rho>1,\ n\in\mathbb{N},\ \forall u\in\mathcal{U},\ \theta\in\{0,\dots,L-1\}^n\ \exists\, v\in\mathcal{U}:\ u^T M_\theta\geq \rho v$

$$\check{s}=\lim_{n\to\infty}\min\left\{\frac{1}{n}\cdot\log\|M_\theta\| : \theta\in[L]^n\right\}$$

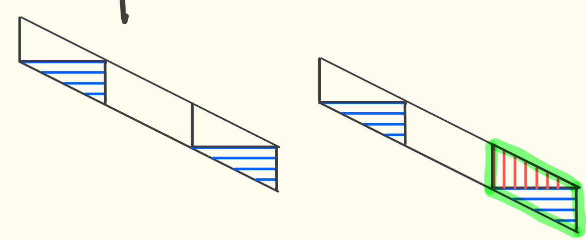
THM. C

B/ Meaning of $\mathcal{U}$ & $u\in\mathcal{U}$:



$$\left.\begin{aligned} u_1&=(1,1,0)\\ u_2&=(0,1,1)\\ u_3&=(1,0,1)\end{aligned}\right\}\ \mathcal{U}=(u_1,u_2,u_3)$$

① $\exists\, u^x\in\{1,\dots,N\},\ \theta^x\in\{0,\dots,L-1\}^{n^x},\ v^x\in\mathcal{U}:\ e_{u^x}' M_{\theta^x}\geq \rho^{n^x} v^x$



② $\exists\, \rho>1,\ n\in\mathbb{N},\ \forall u\in\mathcal{U},\ \theta\in\{0,\dots,L-1\}^n\ \exists\, v\in\mathcal{U}:\ u^T M_\theta\geq \rho v$

Large dev. theory to prove that the # shapes SIMOULTANEOUSLY in all θ grows exponentially.